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Artificial Boundary Conditions for the Linearized Compressible Navier–Stokes Equations

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The compressible Navier–Stokes equations belong to the class of *incompletely parabolic* systems. The general method developed by Laurence Halpern for deriving artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems is applied to the linearized compressible Navier–Stokes equations to obtain high order artificial boundary conditions which are valid for small viscosities, high time frequencies and long space wavelengths. They are implemented in 1D and 2D model problems and compared to the most commonly used boundary conditions to validate the approach, based on asymptotic expansions with respect to the viscosity. The "improved artificial boundary conditions of order (1,1)" provide the best results.

1. INTRODUCTION

In order to compute in a bounded region a flow modeled by a problem formulated on an infinite domain, one often introduces an artificial boundary Γ and tries to write on the domain Ω bounded by Γ an initial boundary value problem whose solution is as close as possible of the solution of the original problem. When the solution of the mixed problem in Ω coincides with the restriction of the solution of the Cauchy problem, the boundary Γ is said to be *transparent.*

In general, the associated boundary condition, called the *transparent boundary condition,* is integral in time and space on the boundary. For obvious numerical reasons (cpu time and memory requirements), the transparent boundary condition is often replaced by local approximations, i.e., differential in time and space, the

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artificial boundary conditions. However, computer limitations do not absolutely necessitate the localization of the transparent (i.e., exact) boundary condition. In fact, different techniques have been proposed in the literature to effectively implement in practice the discrete counterparts to the pseudodifferential operators that are involved in the formulation of the transparent boundary condition on the continuous level. The corresponding approaches apply to computation of both inviscid [25–30] and viscous [31–33] flows.

The artificial boundary conditions are required to give rise to well-posed mixed problems whose solutions are "good" approximations of the initial problem, thus allowing us to place the artificial boundaries as close as possible to the region of interest. Up to now, the most employed method consists in putting the artificial boundary away of the boundary layer and in writing artificial boundary conditions for the compressible unsteady Euler equations. The subsonic case appears to be the most interesting because in 2D, for example, three conditions have to be specified at inflow and one at outflow, whereas in the supersonic case all the variables must be specified at inflow and none at outflow.

Two approaches can be distinguished: the linear treatment and the nonlinear treatment. In the linear treatment, the solution outside the artificial boundary is assumed to be a perturbation of a smooth steady state (often constant) about which the equations are linearized. The derivation and the analysis of the artificial boundary conditions are then performed on the linear equations.

Stable boundary conditions are obtained by setting the incoming characteristic variables to zero [16]. Greater accuracy can be achieved through the methods described in [7], where the transparent boundary condition is approximated for waves with normal incidence and high time frequency and in [2] where a far-field approximation of the Green function is used. In both cases, the resulting boundary conditions are differential in time and space and are analyzed by means of the "normal mode analysis."

In the nonlinear treatment, a reasoning on the characteristics [12, 22] provides boundary conditions that are also differential in time and space.

It is well known that the Navier–Stokes equations need more boundary conditions than the Euler equations. Moreover, for slightly viscous flows, the Navier–Stokes equations may be regarded as a perturbation of the Euler equations. In [16], Oliger and Sundström proposed adding extra boundary conditions to those obtained for the Euler equations. In [8], Gustafsson and Sundström complete the boundary conditions for the Euler equations with relations involving normal derivatives and making the energy decrease.

More recently, Abarbanel, Bayliss, and Lustman [1] directly worked on the Navier–Stokes equations, splitting the boundary layer solution into modes and approximating it for low spatial frequencies and under the assumption of small viscosity, already used successfully in [9] for the advection diffusion equation and in [11] for the incompressible Navier–Stokes equations.

In practice, steady flows are computed either by solving the steady equations or by using the pseudo-unsteady approach. In this case, the static pressure is often prescribed at a subsonic outflow but such a boundary condition is known to considerably slow down the convergence to steady state by reflecting spurious waves toward the interior of the computational domain. Rudy and Strikwerda have compared in [20] several boundary conditions for the Euler equations in the case of the flow over a flat plate in order to improve the convergence to steady state. They have also proposed in [19] a nonreflecting condition for subsonic outflow boundaries.

The works of Hagström [34] and of Abarbanel *et al.* [37], Don and Gottlieb [36], Poinsot and Lele [18] on artificial boundary conditions for fluid flow problems are particularly relevant. See also [35].

The compressible Navier–Stokes equations belong to the class of *incompletely parabolic* equations.

Laurence Halpern has developed in [10] a general method for deriving artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems, using the Fourier and Laplace transforms as essential tools after the equations have been linearized about a constant state. The artificial boundary conditions developed herein are valid under the assumptions of small viscosities, high time frequencies, and long space wavelengths.

This method has been applied in [23] to the compressible Navier–Stokes equations, linearized about a constant state, to obtain high order artificial boundary conditions. They have then been implemented and compared, in 1D and 2D model problems, to the most commonly used boundary conditions in order to validate the approach, based on asymptotic expansions with respect to the viscosity.

This article presents the main results of the work reported in Ref. [23] in which the interested reader will find more details. In Section 2, we recall the general method developed in [10] to derive artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems. In Section 3, this method is applied to the 2D compressible Navier–Stokes equations, linearized about a constant state, to derive a hierarchy of artificial boundary conditions. In Section 4, a 1D test case allows for a rigorous study of the approximation with respect to the kinematic viscosity ν, whereas in Section 5, we analyze the effects of approximating the transparent boundary condition with respect to the second parameter $\varepsilon = i\eta/s$ through a 2D model problem.

2. THE GENERAL METHOD

In this part, we recall the general method developed in [10] to derive artificial boundary conditions for incompletely parabolic perturbations of hyperbolic systems. For details, see [10, 21].

2.1. *The Problem to Be Solved*

We consider the system of linear PDEs with constant coefficients

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{N} A^{(j)} \frac{\partial u}{\partial x_j} + \nu \sum_{j,k=1}^{N} P^{(j,k)} \frac{\partial^2 u}{\partial x_j \partial x_k} + F, \quad \text{where } u(x,t) \in \mathbb{R}^n. \tag{2.1}$$

The $P^{(j,k)}$ matrices have the form

$$P^{(j,k)} = \begin{pmatrix} \overline{P}^{(j,k)} & 0 \\ 0 & 0 \end{pmatrix},$$

the block $\overline{P}^{(j,k)} = \overline{P}^{(k,j)}$ being invertible, with rank $\overline{P}^{(j,k)} = r$.

The $A^{(j)}$ matrices admit an analogous decomposition

$$A^{(j)} = \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{pmatrix}$$

We also assume the operators

$$\frac{\partial}{\partial t} - \sum_{j=1}^{N} A^{(j)} \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial t} - \sum_{j=1}^{N} A_{22}^{(j)} \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial t} - \nu \sum_{j,k=1}^{N} \overline{P}^{(j,k)} \frac{\partial^2}{\partial x_j \partial x_k}$$

to be hyperbolic, strictly hyperbolic, and Petrovskii parabolic, respectively.

Under these assumptions, the Cauchy problem associated to (2.1) can be shown to be well posed (see [21]).

The matrix $A^{(1)}$ is assumed to be nonsingular. Its eingenvalues will be denoted $\lambda_1, \ldots, \lambda_n$, with $\lambda_1, \ldots, \lambda_m < 0$ and $\lambda_{m+1}, \ldots, \lambda_n > 0$. The corresponding eingenvectors will be denoted $\Lambda^1, \ldots, \Lambda^n$. For the sake of simplicity, we suppose that $A_{22}^{(1)}$ is a diagonal matrix with p negative eigenvalues.

Finally, we assume that the symbol

$$Q(i\xi; \nu) = \sum_{j=1}^{N} A^{(j)} \xi_j - \nu \sum_{j,k=1}^{N} P^{(j,k)} \xi_j \xi_k \tag{2.2}$$

of the operator

$$Q = \sum_{j=1}^{N} A^{(j)} \frac{\partial}{\partial x_j} + \nu \sum_{j,k=1}^{N} P^{(j,k)} \frac{\partial^2}{\partial x_j \partial x_k} \tag{2.3}$$

is diagonalizable under a transformation analytic in ξ.

Remark 2.1. All the above assumptions are fulfilled by the compressible Navier–Stokes equations when linearized about a constant state. The left half-space $\{x \in \mathbb{R}^N;\, x_1 < 0\}$ will be denoted $\mathbb{R}^N_-$. We intend to write a boundary condition on $\Gamma = \partial(\mathbb{R}^N_-)$, guaranteeing that the solution of the associated mixed problem in $\mathbb{R}^N_-$ is the restriction to $\mathbb{R}^N_-$ of the solution of the Cauchy problem, the so-called transparent boundary condition.

2.2. *The Transmission Conditions*

We consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \sum\limits_{j=1}^{N} A^{(j)} \dfrac{\partial u}{\partial x_j} + \nu \sum\limits_{j,k=1}^{N} P^{(j,k)} \dfrac{\partial^2 u}{\partial x_j \partial x_k} + F, \quad x \in \mathbb{R}^N, t > 0, \\ u(.,0) = u_0 \end{cases} \tag{2.4}$$

where F and u_0 have compact support in the left-half space.

A variational formulation shows (see [10]) that u satisfies (2.4) if and only if its restrictions u^- and u^+ to the left and right half-spaces, respectively, solve

$$\begin{cases} \dfrac{\partial u^-}{\partial t} - Qu^- = F, \quad x \in \mathbb{R}^N_-, t > 0, \\ u^-(.,0) = u_0 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u^+}{\partial t} - Qu^+ = 0, \quad x \in \mathbb{R}^N_+, t > 0, \\ u^+(.,0) = 0 \end{cases}$$

with the *transmission boundary conditions* through Γ

$$\sum_{j=1}^{N} \overline{P}^{(j,1)} \frac{\partial u^{-I}}{\partial x_j} = \sum_{j=1}^{N} \overline{P}^{(j,1)} \frac{\partial u^{+I}}{\partial x_j}. \tag{2.5}$$

$$u^- = u^+ \tag{2.6}$$

u^I denotes the vector formed with the first r components of u.

2.3. *The Transparent Boundary Condition and the Method of Approximation*

By use of the transmission condition (2.6), it is possible to explicitly express u^+ as a function of u^-. Introducing this expression in (2.5), we obtain the transparent boundary condition. More precisely, let us consider the initial boundary value problem on $\mathbb{R}^N_+$:

$$\begin{cases} \dfrac{\partial u^+}{\partial t} - Qu^+ = 0, \qquad x \in \mathbb{R}^N_+, t > 0, \\ u^+(.,0) = 0 \\ \begin{pmatrix} u_1^+ \\ \vdots \\ u_{r+p}^+ \end{pmatrix} = \begin{pmatrix} u_1^- \\ \vdots \\ u_{r+p}^- \end{pmatrix}, \quad x \in \Gamma, t > 0. \end{cases}$$

The solution of this strongly well-posed problem reads (see [10])

$$\widehat{u^+}(x_1, \eta, s) = \sum_{i=1}^{r+p} \lambda_i e^{\xi_i x_1} \Phi^i, \tag{2.7}$$

where $\widehat{u^+}(x_1, \eta, s)$ is the Fourier–Laplace transform of $u^+(x_1, ., .)$ at (η, s), with $\mathrm{Re}(s) > 0$, variables η and s corresponding to $y = (x_2, ..., x_N)$ and t, respectively. (ξ_i, Φ^i) is the solution of

$$(Q(\xi, i\eta; \nu) - sI)\Phi = 0 \tag{2.8}$$

with $\mathrm{Re}(\xi_i) < 0$.

As a matter of fact, it can be shown when $\mathrm{Re}(s) > 0$ that among the $n + r$ solutions ξ of (2.8), $r + p$ have a strictly negative real part, while the $n - p$ left have a strictly positive real part (see [10]).

The general solution of the Fourier–Laplace transform of $\partial u^+/\partial t - Qu^+ = 0$ with respect to y and t reads

$$\widehat{u^+} = \sum \lambda_i e^{\xi_i x_1} \Phi^i,$$

where (ξ_i, Φ^i) are given by (2.8), since we have supposed that the symbol of Q was diagonalizable. In order for $\widehat{u^+}$ to be in L^2, the coefficient λ_i must vanish when $\mathrm{Re}(\xi_i) \geq 0$ and we obtain the expression in (2.7).

For (ξ, Φ) verifying (2.8), ξ and Φ will be denoted from now on as the "generalized eigenvalue" and "generalized eigenvector" (associated to the generalized eigenvalue ξ), respectively. The generalized eigenvalues and eigenvectors are functions of η, s, and ν.

The *transparent boundary condition* on Γ for the negative half-space reads:

$$\nu \frac{d\hat{u}^I}{dx_1} = \sum_{j=1}^{r+p} \hat{u}_j \sum_{i=1}^{r+p} \nu \xi_i M_{ij}^{-1} \Phi^{i^I} \tag{2.9}$$

$$\hat{u}_k = \sum_{j=1}^{r+p} \hat{u}_j \sum_{i=1}^{r+p} M_{ij}^{-1} \Phi^i_k, \quad k = r + p + 1, ..., n. \tag{2.10}$$

The $(r + p, r + p)$ matrix M is defined by

$$M_{ij} = \Phi^j_i \tag{2.11}$$

and M^{-1} is the inverse of M. For the sake of simplicity, the notation M_{ij}^{-1} is used instead of $(M^{-1})_{ij}$ for the entrees of M^{-1} since the inverse entrees of the matrix M are never used.

As we have assumed that the symbol of Q was diagonalizable, M is a nonsingular matrix and the λ_i's are given by the boundary conditions $\sum_{i=1}^{r+p} \lambda_i \Phi^i_j = \hat{u}_j$, $j = 1, ..., r + p$: $\lambda_i = \sum_{j=1}^{r+p} M_{ij}^{-1} \hat{u}_j$.

It contains $n - p$ scalar conditions, where n is the number of unknowns and p is the number of negative eigenvalues of $A_{22}^{(1)}$. As the general eigenvalues and eigenvectors are nonrational functions of η and s, the transparent boundary condition is integral with respect to y and t.

We assume from now on that $\nu \ll 1$. At first, the transparent boundary condition is approximated with respect to the parameter ν through asymptotic expansions of the generalised eigenvalues and eigenvectors (when Re $s > 0$, it can be shown that r generalized eigenvalues tend to infinity as $1/\nu$, the other having a finite limit. That is why condition (2.9) contains $\nu\,\xi_i$, instead of ξ_i, in order for expression $\sum_{j=1}^{r+p} \hat{u}_j \sum_{i=1}^{r+p} \xi_i\, M_{ij}^{-1}\, \Phi^{i^l}$ to have a regular asymptotic behaviour when ν tends to 0). We thus obtain a condition that remains integral with respect to time and space, but which can be made differential following the lines drawn by Engquist and Majda in [7] for hyperbolic problems that consist in asymptotic expansions with respect to $i\eta/s \ll 1$.

3. APPLICATION TO THE LINEARIZED COMPRESSIBLE NAVIER–STOKES EQUATIONS IN TWO SPACE DIMENSIONS

In two space dimensions, the compressible Navier–Stokes equations read

$$\rho \frac{dV_i}{dt} = -\frac{\partial p}{\partial x_i} + \mu \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\mathrm{div}\,\mathbf{V}\right), \quad i = 1, 2 \tag{3.1}$$

$$\rho \frac{d(C_v T)}{dt} = -p\,\mathrm{div}\,\mathbf{V} + \mu \sum_{j=1}^{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\mathrm{div}\,\mathbf{V}\right) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) \tag{3.2}$$

$$\frac{d\rho}{dt} = -\,p\,\mathrm{div}\,\mathbf{V}, \tag{3.3}$$

where d/dt is the advection operator, $\partial/\partial t + V_1\,(\partial/\partial x_1) + V_2\,(\partial/\partial x_2)$, ρ, V_1, V_2, p, T, μ, and k denote the density, the components of the velocity vector $\mathbf{V}$, the pressure, the temperature, the viscosity, and the thermal conductivity, respectively. We have assumed $3\lambda + 2\mu = 0$ (Stokes' hypothesis). Pressure, density, and temperature are related through the state equation for an ideal gas $p = \rho R T$ with R the Mayer's constant.

We assume the flow in the positive half-space to be a small perturbation of a constant state $(\overline{\mathbf{V}}, \overline{T}, \overline{\rho})$:

$$\mathbf{V} = \overline{\mathbf{V}} + \tilde{\mathbf{V}}$$

$$T = \overline{T} + \tilde{T}$$

$$\rho = \overline{\rho} + \tilde{\rho}.$$

Linearizing (3.1), (3.2), and (3.3) around $(\overline{\mathbf{V}}, \overline{T}, \overline{\rho})$, we obtain the incompletely parabolic system,

$$\frac{\partial u}{\partial t} = A^{(1)}\frac{\partial u}{\partial x_1} + A^{(2)}\frac{\partial u}{\partial x_2} + \nu \sum_{j,k=1}^{2} P^{j,k}\frac{\partial^2 u}{\partial x_j \partial x_k}, \tag{3.4}$$

where $u = (\tilde{V}_1, \tilde{V}_2, \tilde{T}, \tilde{\rho}/\overline{\rho})^t$, $\nu = \mu/\overline{\rho}$ and

$$A^{(1)} = \begin{pmatrix} -\overline{V}_1 & 0 & -R & -R\overline{T} \\ 0 & -\overline{V}_1 & 0 & 0 \\ -(\gamma-1)\overline{T} & 0 & -\overline{V}_1 & 0 \\ -1 & 0 & 0 & -\overline{V}_1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} -\overline{V}_2 & 0 & 0 & 0 \\ 0 & -\overline{V}_2 & -R & -R\overline{T} \\ 0 & -(\gamma-1)\overline{T} & -\overline{V}_2 & 0 \\ 0 & -1 & 0 & -\overline{V}_2 \end{pmatrix}$$

$$P^{(1,1)} = \mathrm{diag}(4/3, 1, \gamma/\mathrm{Pr}, 0), \quad P^{(2,2)} = \mathrm{diag}(1, 4/3, \gamma/\mathrm{Pr}, 0)$$

$$P^{(1,2)} = P^{(2,1)} = \begin{pmatrix} 0 & 1/6 & 0 & 0 \\ 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As matrix $A_{22}^{(1)}$ (see 2.1) reduces to $(-\overline{V}_1)$, the number of boundary conditions is three for an outflow boundary ($\overline{V}_1 > 0$) and four for an inflow boundary ($\overline{V}_1 < 0$).

If we multiply matrix $Q(\xi, i\eta; \nu) - sI$ by ν, we get $\nu[Q(\xi, i\eta; \nu) - sI] = Q(\zeta, \nu s\varepsilon; 1) - \nu sI$ with $\zeta = \nu\xi$ and $\varepsilon = i\eta/s$. In the sequel, ζ will (abusively) be referred to as a "generalized eigenvalue."

In order to approximate the transparent boundary condition, each generalized eigenvalue ζ and its related generalized eigenvector Φ will be expanded to the order o_ν with respect to parameter ν. Each term of this development will in turn be approximated to the order o_ε with respect to ε. The resulting boundary conditions will be denoted "artificial boundary conditions of order (o_ν, o_ε)." Their expression will be given in both cases $(o_\nu, o_\varepsilon) = (1, 0)$ and $(1, 1)$.

3.1. *Generalized Eigenvalues and Eigenvectors*

We use the notations

$$\zeta = \zeta_0 + \nu\zeta_1 + \nu^2\zeta_2 + O(\nu^3) \tag{3.5}$$

$$\Phi = \Phi_0 + \nu\Phi_1 + O(\nu^2), \tag{3.6}$$

where ζ_0, ζ_1, ζ_2, Φ_0, and Φ_1 are functions of s and ε.

Substituting expansion (3.5) in the expression of matrix $Q(\zeta, \varepsilon\nu s; 1)$ gives

$$\begin{aligned} Q(\zeta, \varepsilon\nu s; 1) - \nu sI = {} & \chi_0(A^{(1)} + \chi_0 P^{(1,1)}) + \nu s\,[\chi_1(A^{(1)} + 2\chi_0 P^{(1,1)}) \\ & + \varepsilon(A^{(2)} + 2\chi_0 P^{(1,2)}) - I] + \nu^2 s^2[\chi_2(A^{(1)} + 2\chi_0 P^{(1,1)}) \\ & + \chi_1^2(P^{(1,1)} + 2\varepsilon\chi_1 P^{(1,2)}) + \varepsilon^2 P^{(2,2)}] + O(\nu^3), \end{aligned}$$

where $\chi_i = \zeta_i/s^i$, $i = 0, 1, 2$ (in s^i, i is a power and not a superscript).

Making ν tend to zero in equation $\det[Q(\zeta, \nu s\varepsilon; 1) - \nu sI] = 0$, we obtain, by continuity with respect to ν: $\chi_0^4 \det(A^{(1)} + \chi_0 P^{(1,1)}) = 0$. Thus, we have to distinguish the bounded generalized eigenvalues ($\chi_0 = 0$) from those tending to infinity as $1/\nu$.

We obtain the asymptotic expansions

$$\zeta^{(1)} = \frac{\nu s}{-\overline{V}_1 - \overline{C}}(1 + \varepsilon\overline{V}_2) + \frac{(\nu s)^2}{2(\overline{V}_1 - \overline{C})^3}\left(\frac{4}{3} + \frac{\gamma - 1}{\Pr}\right)(1 + 2\varepsilon\overline{V}_2) + O(\nu^3) + O(\varepsilon^2) \tag{3.7}$$

$$\zeta^{(2)} = \frac{\nu s}{-\overline{V}_1}(1 + \varepsilon\overline{V}_2) + \frac{(\nu s)^2}{\overline{V}_1^3}(1 + 2\varepsilon\overline{V}_2) + O(\nu^3) + O(\varepsilon^2) \tag{3.8}$$

$$\zeta^{(3)} = \frac{\nu s}{-\overline{V}_1}(1 + \varepsilon\overline{V}_2) + \frac{(\nu s)^2}{\Pr\overline{V}_1^3}(1 + 2\varepsilon\overline{V}_2) + O(\nu^3) + O(\varepsilon^2) \tag{3.9}$$

$$\zeta^{(4)} = \frac{\nu s}{-\overline{V}_1 + \overline{C}}(1 + \varepsilon\overline{V}_2) + \frac{(\nu s)^2}{2(\overline{V}_1 - \overline{C})^3}\left(\frac{4}{3} + \frac{\gamma - 1}{\Pr}\right) + O(\nu^3) + O(\varepsilon^2) \tag{3.10}$$

for the generalized eigenvalues ζ such that $\chi_0 = 0$,

$$\Phi^1 = \begin{pmatrix} \overline{C} \\ 0 \\ (\gamma-1)\overline{T} \\ 1 \end{pmatrix} + \varepsilon\overline{C}(-\overline{V}_1 - \overline{C})\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \nu s\frac{1}{\overline{V}_1 + \overline{C}}\begin{pmatrix} \dfrac{\frac{4}{3} + \frac{\gamma-1}{\Pr}}{2} \\ 0 \\ \dfrac{\gamma(\gamma-1)\overline{T}}{\Pr\overline{C}} \\ 0 \end{pmatrix}$$
$$+ \nu s\varepsilon\frac{1}{\overline{V}_1 + \overline{C}}\begin{pmatrix} \dfrac{\frac{4}{3} + \frac{\gamma-1}{\Pr}}{2}\overline{V}_2 \\ -\dfrac{\frac{4}{3} + \frac{\gamma-1}{\Pr}}{2}(\overline{V}_1 + 2\overline{C}) \\ \dfrac{\gamma(\gamma-1)\overline{T}}{\Pr\overline{C}}\overline{V}_2 \\ 0 \end{pmatrix} + O(\nu^2) + O(\varepsilon^2) \tag{3.11}$$

$$\Phi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \varepsilon\overline{V}_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu s\mathbf{0} + \nu s\varepsilon\frac{\overline{V}_1^2 + \overline{C}^2}{\overline{V}_1\overline{C}^2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + O(\nu^2) + O(\varepsilon^2) \tag{3.12}$$

$$\Phi^3 = \begin{pmatrix} 0 \\ 0 \\ \overline{T} \\ -1 \end{pmatrix} + \varepsilon\mathbf{0} + \nu s\frac{-1}{\Pr\overline{V}_1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu s\varepsilon\frac{-\overline{V}_2}{\overline{V}_1\Pr}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + O(\nu^2) + O(\varepsilon^2) \tag{3.13}$$

$$\Phi^4 = \begin{pmatrix} \overline{C} \\ 0 \\ -(\gamma-1)\overline{T} \\ -1 \end{pmatrix} + \varepsilon\overline{C}(-\overline{V}_1 + \overline{C})\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \nu s \frac{1}{\overline{V}_1 - \overline{C}} \begin{pmatrix} -\dfrac{\dfrac{4}{3} + \dfrac{\gamma-1}{\mathrm{Pr}}}{2} \\ 0 \\ \dfrac{\gamma(\gamma-1)\overline{T}}{\mathrm{Pr}\,\overline{C}} \\ 0 \end{pmatrix}$$

$$+ \nu s \varepsilon \frac{1}{\overline{V}_1 - \overline{C}} \begin{pmatrix} -\dfrac{\dfrac{4}{3} + \dfrac{\gamma-1}{\mathrm{Pr}}}{2}\overline{V}_2 \\ \dfrac{\dfrac{4}{3} + \dfrac{\gamma-1}{\mathrm{Pr}}}{2}(\overline{V}_1 - 2\overline{C}) \\ \dfrac{\gamma(\gamma-1)\overline{T}}{\mathrm{Pr}\,\overline{C}}\overline{V}_2 \\ 0 \end{pmatrix} + O(\nu^2) + O(\varepsilon^2) \qquad (3.14)$$

for the corresponding generalized eigenvectors Φ, with $\overline{C} = (\gamma R\overline{T})^{1/2}$ (linearized sound speed),

$$\zeta^{(5)} = \theta_1 + \frac{\nu s}{\theta_1} \frac{\left(\dfrac{4}{3} + \dfrac{\gamma + \dfrac{\overline{C}^2}{\overline{V}_1^2}}{\mathrm{Pr}}\right)\theta_1 - 2\overline{V}_1}{\dfrac{4}{3}\dfrac{\gamma}{\mathrm{Pr}}(\theta_1 - \theta_2)}(1 + \varepsilon\overline{V}_2) + O_\varepsilon(\nu^2) \qquad (3.15)$$

$$\zeta^{(6)} = \theta_2 + \frac{\nu s}{\theta_2} \frac{\left(\dfrac{4}{3} + \dfrac{\gamma + \dfrac{\overline{C}^2}{\overline{V}_1^2}}{\mathrm{Pr}}\right)\theta_2 - 2\overline{V}_1}{\dfrac{4}{3}\dfrac{\gamma}{\mathrm{Pr}}(\theta_2 - \theta_1)}(1 + \varepsilon\overline{V}_2) + O_\varepsilon(\nu^2) \qquad (3.16)$$

$$\zeta^{(7)} = \overline{V}_1 + \frac{\nu s}{\overline{V}_1}(1 + \varepsilon\overline{V}_2) + O_\varepsilon(\nu^2) \qquad (3.17)$$

for the generalized eigenvalues ζ, such that $\chi_0 \neq 0$ and

$$\Phi^5 = \begin{pmatrix} \overline{V}_1 \\ 0 \\ U_1 \\ -1 \end{pmatrix} + \nu s \frac{1}{\theta_1 \overline{V}_1} \begin{pmatrix} 0 \\ 0 \\ U_1 \dfrac{\dfrac{4}{3}(\theta_1 + \theta_2) - 2\overline{V}_1}{\dfrac{4}{3}(\theta_1 - \theta_2)} \\ 1 \end{pmatrix}(1 + \varepsilon\overline{V}_2) + O_\varepsilon(\nu^2) \qquad (3.18)$$

$$\Phi^6 = \begin{pmatrix} \overline{V}_1 \\ 0 \\ U_2 \\ -1 \end{pmatrix} + \nu s \frac{1}{\theta_2 \overline{V}_1} \begin{pmatrix} 0 \\ 0 \\ U_2 \dfrac{\frac{4}{3}(\theta_1 + \theta_2) - 2\overline{V}_1}{\frac{4}{3}(\theta_2 - \theta_1)} \\ 1 \end{pmatrix} (1 + \varepsilon \overline{V}_2) + O_\varepsilon(\nu^2) \tag{3.19}$$

$$\Phi^7 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \nu s \varepsilon \begin{pmatrix} -\dfrac{1}{\overline{V}_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} + O_\varepsilon(\nu^2) \tag{3.20}$$

for the associated generalized eigenvectors Φ. The derivation of expansions (3.7)–(3.20) is delineated in [23].

Remark 3.1. In expansions (3.15)–(3.20) the notation $O_\varepsilon(\nu^2)$ indicates that the residual is a function of ε. θ_1 and θ_2 are the roots of the second-order algebraic equation with respect to θ,

$$-\overline{V}_1 \left(\frac{4}{3}\theta - \overline{V}_1\right)\left(\frac{\gamma}{\mathrm{Pr}}\theta - \overline{V}_1\right) - \overline{C}^2 \left(\frac{1}{\mathrm{Pr}}\theta - \overline{V}_1\right) = 0 \tag{3.21}$$

with convention $\theta_1 < \theta_2$ and U_j, $j = 1, 2$, is defined by

$$U_j = \frac{(\gamma - 1)\overline{T}\,\overline{V}_1}{\frac{\gamma}{\mathrm{Pr}}\theta_j - \overline{V}_1}.$$

It is shown in [23] that Eq. (3.21) always admits two distinct real roots.

We are now able to derive the artificial boundary conditions of order (1, 0) and (1, 1).

3.2. *General Rules for the Derivation of the Artificial Boundary Conditions*

We introduce the $(r + p, r + p)$ matrices M_{00}, M_{01}, M_{10}, and M_{11} defined as

$$(M_{00})_{ij} = (\Psi^j_{00})_i, \quad (M_{01})_{ij} = (\Psi^j_{01})_i, \quad (M_{10})_{ij} = (\Psi^j_{10})_i,$$
$$(M_{11})_{ij} = (\Psi^j_{11})_i, \quad (\Psi^j = \Phi^j/s^j).$$

In the notation $\Psi_{\alpha\beta}$, the subscripts α and β are the orders with respect to ν and ε, respectively.

Matrix M defined by (2.11) then has the asymptotic expansion

$$M = M_{00} + \varepsilon M_{01} + \nu s(M_{10} + \varepsilon M_{11}) + O(\nu^2) + O(\varepsilon^2).$$

For its inverse, we have $M^{-1} = (M^{-1})_{00} + \varepsilon(M^{-1})_{01} + \nu s[(M^{-1})_{10} + \varepsilon(M^{-1})_{11})] + O(\nu^2) + O(\varepsilon^2)$ with

$$(M^{-1})_{00} = M_{00}^{-1}, \quad (M^{-1})_{01} = -M_{00}^{-1} M_{01} M_{00}^{-1}, \quad (M^{-1})_{10} = -M_{00}^{-1} M_{10} M_{00}^{-1},$$
$$(M^{-1})_{11} = M_{00}^{-1} (M_{01} M_{00}^{-1} M_{10} + M_{10} M_{00}^{-1} M_{01} - M_{11}) M_{00}^{-1}.$$

For a bounded generalized eigenvalue ζ_i, as $(\zeta_i)_0 = 0$ we have

$$\begin{aligned}\zeta_i M_{ij}^{-1} \Phi^{i^I} = {} & \nu s(\zeta_i)_{10}(M_{ij}^{-1})_{00}\Phi_{00}^{i^I} + \nu s\varepsilon[(\zeta_i)_{11}(M_{ij}^{-1})_{00}\Phi_{00}^{i^I} \\ & + (\zeta_i)_{10}(M_{ij}^{-1})_{01}\Phi_{00}^{i^I} + (\zeta_i)_{10}(M_{ij}^{-1})_{00}\Phi_{01}^{i^I}] + O(\nu^2) + O(\varepsilon^2)\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}M_{ij}^{-1}\,\Phi_k^i = {} & (M_{ij}^{-1})_{00}(\Phi_k^i)_{00} \\ & + \varepsilon[(M_{ij}^{-1})_{01}\,(\Phi_k^i)_{00} + (M_{ij}^{-1})_{00}(\Phi_k^i)_{01}] \\ & + \nu s[(M_{ij}^{-1})_{10}(\Phi_k^i)_{00} + (M_{ij}^{-1})_{00}(\Phi_k^i)_{10}] \\ & + \nu s\varepsilon[(M_{ij}^{-1})_{11}(\Phi_k^i)_{00} + (M_{ij}^{-1})_{00}(\Phi_k^i)_{11} \\ & + (M_{ij}^{-1})_{10}(\Phi_k^i)_{01} + (M_{ij}^{-1})_{01}(\Phi_k^i)_{10}]O(\nu^2) + O(\varepsilon^2)\end{aligned} \tag{3.23}$$

whereas for an unbounded generalized eigenvalue ζ_i, we have

$$\begin{aligned}\zeta_i M_{ij}^{-1}\Phi^{i^I} = {} & (\zeta_i)_0(M_{ij}^{-1})_{00}\Phi_0^{i^I} + \varepsilon(\zeta_i)_0(M_{ij}^{-1})_{01}\Phi_0^{i^I} \\ & + \nu s[(\zeta_i)_0(M_{ij}^{-1})_{00}\Phi_0^{i^I} + (\zeta_i)_0(M_{ij}^{-1})_{10}\Phi_0^{i^I} + (\zeta_i)_0(M_{ij}^{-1})_{10}\Phi_{10}^{i^I}] \\ & + \nu s\varepsilon[(\zeta_i)_{11}(M_{ij}^{-1})_{00}\Phi_0^{i^I} + (\zeta_i)_0(M_{ij}^{-1})_{11}\Phi_0^{i^I} + (\zeta_i)_0(M_{ij}^{-1})_{00}\Phi_{11}^{i} \\ & + (\zeta_i)_{10}(M_{ij}^{-1})_{01}\Phi_0^{i^I} + (\zeta_i)_0(M_{ij}^{-1})_{01}\Phi_{10}^{i^I}] + O(\nu^2) + O(\varepsilon^2)\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}M_{ij}^{-1}\,\Phi_k^i = {} & (M_{ij}^{-1})_{00}(\Phi_k^i)_0 + \varepsilon(M_{ij}^{-1})_{01}\,(\Phi_k^i)_0 \\ & + \nu s[(M_{ij}^{-1})_{10}(\Phi_k^i)_0 + (M_{ij}^{-1})_{00}(\Phi_k^i)_{10}] \\ & + \nu s\varepsilon[(M_{ij}^{-1})_{11}(\Phi_k^i)_0 + (M_{ij}^{-1})_{00}(\Phi_k^i)_{11} + (M_{ij}^{-1})_{01}(\Phi_k^i)_{10}] \\ & + O(\nu^2) + O(\varepsilon^2).\end{aligned} \tag{3.25}$$

Let A be the $(r, r+p)$ matrix with jth column given by $\sum_{i=1}^{r+p} \zeta_i M_{ij}^{-1}\Phi^{i^I}$ and B (if it exists) the $(n - (r + p), r + p)$ matrix whose jth column is expressed by $\sum_{i=1}^{r+p} M_{ij}^{-1} \begin{pmatrix} \Phi_{r+p+1}^i \\ \vdots \\ \Phi_n^i \end{pmatrix}$. To formulas (3.22)–(3.25) correspond the derivations

$$A = A_{00} + \varepsilon A_{01} + \nu s A_{10} + \nu s \varepsilon A_{11} + O(\nu^2) + O(\varepsilon^2)$$
$$B = B_{00} + \varepsilon B_{01} + \nu s B_{10} + \nu s \varepsilon B_{11} + O(\nu^2) + O(\varepsilon^2).$$

In Sections 3.3 and 3.4, we will consider four cases:

* *Supersonic inflow* $(\overline{V}_1 < -\overline{C})$. $p = 0$, $r + p = 3$, $\theta_1 < \theta_2 < 0$. The generalized eigenvalues ζ_i, $i = 1, ..., r + p$ are given by expansions (3.15), (3.16), and (3.17), respectively.

* *Subsonic inflow* $(-\overline{C} < \overline{V}_1 < 0)$. $p = 0$, $r + p = 3$, $\theta_1 < 0 < \theta_2$. The generalized eigenvalues $\zeta_i, i = 1, ..., r + p$ are given by expansions (3.7), (3.15), and (3.17), respectively.

* *Subsonic outflow* $(0 < \overline{V}_1 < \overline{C})$. $p = 1$, $r + p = 4$, $\theta_1 < 0 < \theta_2$. The generalized eigenvalues ζ_i, $i = 1, ..., r + p$ are given by expansions (3.7), (3.8), (3.9), and (3.15), respectively.

* *Supersonic outflow* $(\overline{C} < \overline{V}_1)$. $p = 1$, $r + p = 4$, $0 < \theta_1 < \theta_2$. The generalized eigenvalues ζ_i, $i = 1, ..., r + p$ are given by expansions (3.7), (3.8), (3.9), and (3.10), respectively.

For an inflow boundary $(\overline{V}_1 < 0)$, A is a (3, 3) matrix and B is a (1, 3) matrix. In terms of Fourier–Laplace variables, the artificial boundary conditions have the general form

$$\nu \frac{d}{dx_1} \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} = (A_{00} + \nu s\, A_{10}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix}$$
$$\frac{\hat{\tilde{\rho}}}{\overline{\rho}} = (B_{00} + \nu s\, B_{10}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} \tag{3.26}$$

for the order (1, 0),

$$\nu \frac{d}{dx_1} \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} = (A_{00} + \varepsilon\, A_{01} + \nu s\, A_{10} + \nu s \varepsilon\, A_{11}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix}$$
$$\frac{\hat{\tilde{\rho}}}{\overline{\rho}} = (B_{00} + \varepsilon\, B_{01} + \nu s\, B_{10} + \nu s \varepsilon\, B_{11}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} \tag{3.27}$$

for the order (1, 1).

For an outflow boundary ($\overline{V}_1 > 0$), A is a (3, 4) matrix and B does not exist. The artificial boundary conditions have the general form

$$\nu \frac{d}{dx_1}\begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} = (A_{00} + \nu s\, A_{10}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \\ \hat{\tilde{\rho}}/\overline{\rho} \end{pmatrix} \tag{3.28}$$

for the order (1, 0) and

$$\nu \frac{d}{dx_1}\begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \end{pmatrix} = (A_{00} + \varepsilon\, A_{01} + \nu s\, A_{10} + \nu s \varepsilon\, A_{11}) \begin{pmatrix} \hat{\tilde{V}}_1 \\ \hat{\tilde{V}}_2 \\ \hat{\tilde{T}} \\ \hat{\tilde{\rho}}/\overline{\rho} \end{pmatrix} \tag{3.29}$$

for the order (1, 1).

3.3. *Artificial Boundary Conditions of Order* (1, 0)

For an inflow boundary, an inverse Fourier–Laplace transform of conditions (3.26) gives

$$\nu \frac{\partial}{\partial x_1}\begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix} = \left(A_{00} + \nu\, A_{10} \frac{\partial}{\partial t}\right) \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix}$$

$$\frac{\tilde{\rho}}{\overline{\rho}} = \left(B_{00} + \nu\, B_{10} \frac{\partial}{\partial t}\right) \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix},$$

whereas for an outflow boundary, we obtain

$$\nu \frac{\partial}{\partial x_1}\begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix} = \left(A_{00} + \nu\, A_{10} \frac{\partial}{\partial t}\right) \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \\ \tilde{\rho}/\overline{\rho} \end{pmatrix}.$$

3.4. *Artificial Boundary Conditions of Order* (1, 1)

If the ith line of matrix A_{01} is not equal to zero, the corresponding boundary condition has to be multiplied by s before applying the inverse Fourier–Laplace transform and we get

$$\nu \frac{\partial^2}{\partial x_1 \partial t} u_i = \left[(A_{i.})_{00} \frac{\partial}{\partial t} + (A_{i.})_{01} \frac{\partial}{\partial x_2} + \nu (A_{i.})_{10} \frac{\partial^2}{\partial t^2} + \nu (A_{i.})_{11} \frac{\partial^2}{\partial x_2 \partial t} \right] \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix}$$

for an inflow boundary,

$$\nu \frac{\partial^2}{\partial x_1 \partial t} u_i = \left[(A_{i.})_{00} \frac{\partial}{\partial t} + (A_{i.})_{01} \frac{\partial}{\partial x_2} + \nu (A_{i.})_{10} \frac{\partial^2}{\partial t^2} + \nu (A_{i.})_{11} \frac{\partial^2}{\partial x_2 \partial t} \right] \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \\ \tilde{\rho}/\overline{\rho} \end{pmatrix}$$

for an outflow boundary.

If the ith line of matrix A_{01} is equal to zero, we simply get

$$\nu \frac{\partial}{\partial x_1} u_i = \left[(A_{i.})_{00} + \nu (A_{i.})_{10} \frac{\partial}{\partial t} + \nu (A_{i.})_{11} \frac{\partial}{\partial x_2} \right] \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix}$$

for an inflow boundary,

$$\nu \frac{\partial}{\partial x_1} u_i = \left[(A_{i.})_{00} + \nu (A_{i.})_{10} \frac{\partial}{\partial t} + \nu (A_{i.})_{11} \frac{\partial}{\partial x_2} \right] \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \\ \tilde{\rho}/\overline{\rho} \end{pmatrix}$$

for an outflow boundary.

In the same way, when matrix B_{01} is not equal to zero, the hyperbolic condition reads

$$\frac{\partial}{\partial t} \frac{\tilde{\rho}}{\overline{\rho}} = \left[B_{00} \frac{\partial}{\partial t} + B_{01} \frac{\partial}{\partial x_2} + \nu B_{10} \frac{\partial^2}{\partial t^2} + \nu B_{11} \frac{\partial^2}{\partial x_2 \partial t} \right] \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{T} \end{pmatrix},$$

else we only have

$$\frac{\tilde{\rho}}{\overline{\rho}} = \left(B_{00} + \nu B_{10}\frac{\partial}{\partial t} + \nu B_{11}\frac{\partial}{\partial x_2}\right)\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{T}\end{pmatrix}.$$

For the supersonic inflow case $A_{01} = 0$ and $B_{01} = 0$. For the supersonic outflow case, $A_{01} = 0$. Thus, in the supersonic case, the conditions of order (1, 1) are obtained by addition of terms in $\nu(\partial/\partial x_2)$ to conditions of order (1, 0).

The expression of the artificial boundary conditions of order $(1, 1)$ is given below for the two subsonic cases. Because matrices A_{11} and B_{11} are very complicated, we have preferred not to give their expressions. In practice, they are evaluated by the code using asymptotic expansions (3.22)–(3.25) (see Remark 3.2 below).

* *The subsonic inflow case:*

$$\nu\frac{\partial\tilde{V}_1}{\partial x_1} = \frac{\theta_1\left(\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1\right)}{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1 - \overline{C}}\left(\tilde{V}_1 - \frac{\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{T}}\right) + \nu\frac{1}{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1 - \overline{C}}\frac{\partial}{\partial t}$$

$$\times\left\{\begin{aligned}&\frac{1}{\overline{V}_1+\overline{C}}\left(\overline{C} + \frac{\theta_1\left(\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1\right)}{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1 - \overline{C}}\times\frac{\frac{4}{3}-\frac{\gamma+1}{\mathrm{Pr}}}{2}\right)\left(\tilde{V}_1 - \frac{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1}{\gamma-1}\frac{\tilde{T}}{\overline{T}}\right) + \left(\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1\right)\\ &\left(\frac{\left(\frac{4}{3}+\frac{\gamma+\frac{\overline{C}^2}{\overline{V}_1^2}}{\mathrm{Pr}}\right)\theta_1 - 2\overline{V}_1}{\frac{\gamma}{\mathrm{Pr}}\theta_1\frac{4}{3}(\theta_1-\theta_2)} + \frac{\overline{C}}{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1 - \overline{C}}\times\frac{\frac{4}{3}(\theta_1+\theta_2) - 2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)\overline{V}_1}\right)\\ &\left(\tilde{V}_1 - \frac{\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{T}}\right)\end{aligned}\right\}$$

$$+\,\nu\frac{\partial}{\partial x_2}[(A_{11})_{11}\tilde{V}_1 + (A_{12})_{11}\tilde{V}_2 + (A_{13})_{11}\tilde{T}]; \tag{3.30a}$$

$$\nu\frac{\partial^2\tilde{V}_2}{\partial x_1\partial t} = \overline{V}_1\frac{\partial\tilde{V}_2}{\partial t} + \nu\frac{1}{\overline{V}_1}\frac{\partial^2\tilde{V}_2}{\partial t^2} + \frac{\overline{V}_1\overline{C}(\overline{V}_1+\overline{C})}{\frac{\gamma}{\mathrm{Pr}}\theta_1 - \overline{V}_1 - \overline{C}}\frac{\partial}{\partial x_2}\left(-\tilde{V}_1 + \frac{\overline{V}_1}{U_1}\tilde{T}\right)$$

$$+\,\frac{\partial^2}{\partial x_2\partial t}[(A_{21})_{11}\tilde{V}_1 + (A_{22})_{11}\tilde{V}_2 + (A_{23})_{11}\tilde{T}]; \tag{3.30b}$$

$$\nu \frac{\partial \tilde{T}}{\partial x_1} = \frac{(\gamma-1)\overline{T}\theta_1}{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1 - \overline{C}} \left(\tilde{V}_1 - \frac{\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) + \nu \frac{(\gamma-1)\overline{T}}{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1 - \overline{C}} \frac{\partial}{\partial t}$$

$$\times \left\{ \begin{array}{l} \frac{1}{\overline{V}_1+\overline{C}}\left(1 + \frac{\theta_1}{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1 - \overline{C}} \times \frac{\frac{4}{3} - \frac{\gamma+1}{\Pr}}{2}\right)\left(\tilde{V}_1 - \frac{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) \\ + \left[\frac{\left(\frac{4}{3} + \frac{\gamma + \frac{\overline{C}^2}{\overline{V}_1^2}}{\Pr}\right)\theta_1 - 2\overline{V}_1}{\frac{\gamma}{\Pr}\theta_1 \frac{4}{3}(\theta_1-\theta_2)} + \frac{\frac{4}{3}(\theta_1+\theta_2) - 2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)\overline{V}_1}\left(1+\frac{\overline{C}}{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1 - \overline{C}}\right)\right] \\ \left(\tilde{V}_1 - \frac{\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) \end{array} \right\}$$

$$+ \nu \frac{\partial}{\partial x_2}[(A_{31})_{11}\tilde{V}_1 + (A_{32})_{11}\tilde{V}_2 + (A_{33})_{11}\tilde{T}]; \qquad (3.30c)$$

$$\frac{\tilde{\rho}}{\rho} = \frac{-\left(\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1\right)}{\overline{V}_1\left(\frac{\gamma}{\Pr}\theta_1-\overline{V}_1-\overline{C}\right)}\left(\frac{\frac{\gamma}{\Pr}\theta_1}{\frac{\gamma}{\Pr}\theta_1-\overline{V}_1}\tilde{V}_1 - \frac{\overline{V}_1+\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) - \nu\frac{1}{\left(\frac{\gamma}{\Pr}\theta_1-\overline{V}_1-\overline{C}\right)^2}\frac{\partial}{\partial t}$$

$$\times\left\{ \begin{array}{l} \frac{\frac{\gamma}{\Pr}\theta_1 - \overline{V}_1}{\theta_1}\left[\frac{\frac{4}{3}(\theta_1+\theta_2)-2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)\overline{V}_1}\left(1+\frac{\overline{C}}{\overline{V}_1}\right) - \frac{\frac{\gamma}{\Pr}\theta_1-\overline{V}_1-\overline{C}}{\overline{V}_1^2}\right]\left(\tilde{V}_1 - \frac{\overline{C}}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) \\ + \frac{1}{\overline{V}_1+\overline{C}}\left[\left(\frac{\gamma}{\Pr}\theta_1-\overline{V}_1\right)\left(\frac{\frac{4}{3} - \frac{\gamma+1}{\Pr}}{2\overline{V}_1} - \frac{\frac{\gamma}{\Pr}}{\overline{C}}\right) + \frac{\frac{4}{3}+\frac{\gamma-1}{\Pr}}{2}\right]\left(\tilde{V}_1 - \frac{\frac{\gamma}{\Pr}\theta_1-\overline{V}_1}{\gamma-1}\frac{\tilde{T}}{\overline{\overline{T}}}\right) \end{array} \right\}$$

$$+ \nu\frac{\partial}{\partial x_2}[(B_{11})_{11}\tilde{V}_1+(B_{12})_{11}\tilde{V}_2+(B_{13})_{11}\tilde{T}]. \qquad (3.30d)$$

* *The subsonic outflow case:*

$$\nu\frac{\partial^2\tilde{V}_1}{\partial x_1\partial t} = \frac{\partial}{\partial t}\left\{\frac{\overline{C}\theta_1\overline{V}_1}{Z}\left(\frac{\gamma}{\overline{C}}\tilde{V}_1 - \frac{\tilde{T}}{\overline{\overline{T}}} - \frac{\tilde{\rho}}{\rho}\right) + \nu\frac{1}{\overline{T}Z}\frac{\partial}{\partial t}\right.$$

$$\times\left\{\left[\frac{\gamma\theta_1}{Z}\left(\frac{1}{\Pr} - \frac{\frac{4}{3}-\frac{\gamma-1}{\Pr}}{2}\frac{\overline{V}_1}{\overline{V}_1+\overline{C}}\right) - \frac{\overline{C}}{\overline{V}_1+\overline{C}}\right]\left[(\overline{T} - U_1)\tilde{V}_1 + \overline{V}_1\tilde{T} + \overline{V}_1\overline{T}\frac{\tilde{\rho}}{\rho}\right]\right.$$

$$+\,\overline{V}_1\left[\frac{\left(\frac{4}{3}+\frac{\gamma+\frac{\overline{C}^2}{\overline{V}_1^2}}{\Pr}\right)\theta_1-2\overline{V}_1}{\theta_1\frac{4}{3}\frac{\gamma}{\Pr}(\theta_1-\theta_2)}-\frac{\gamma}{\Pr}\frac{\theta_1}{Z\overline{V}_1}+\frac{\overline{C}}{\overline{V}_1\overline{T}Z}\left(U_1\frac{\frac{4}{3}(\theta_1+\theta_2)-2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)}+\overline{T}\right)\right]$$

$$\left.\left[\gamma\overline{T}\tilde{V}_1-\overline{C}\tilde{T}-\overline{C}\,\overline{T}\frac{\tilde{\rho}}{\overline{\rho}}\right]-\frac{\gamma}{\Pr}\overline{T}\,\theta_1\frac{\tilde{\rho}}{\overline{\rho}}\right\}\right\}-\frac{\gamma\theta_1\overline{V}_1^2}{Z}\frac{\partial}{\partial x_2}\tilde{V}_2$$

$$+\,\nu\frac{\partial^2}{\partial x_2\partial t}\left((A_{11})_{11}\tilde{V}_1+(A_{12})_{11}\tilde{V}_2+(A_{13})_{11}\tilde{T}+(A_{14})_{11}\frac{\tilde{\rho}}{\overline{\rho}}\right];\tag{3.31a}$$

$$\nu\frac{\partial\tilde{V}_2}{\partial x_1}=-\frac{\nu}{\overline{V}_1}\frac{\partial\tilde{V}_2}{\partial t}+\nu\frac{\partial}{\partial x_2}\left[(A_{21})_{11}\tilde{V}_1+(A_{22})_{11}\tilde{V}_2+(A_{23})_{11}\tilde{T}+(A_{24})_{11}\frac{\tilde{\rho}}{\overline{\rho}}\right];\tag{3.31b}$$

$$\nu\frac{\partial^2\tilde{T}}{\partial x_1\partial t}=\frac{\partial}{\partial t}\left\{\frac{\overline{C}\theta_1U_1}{Z}\left(\frac{\gamma}{\overline{C}}\tilde{V}_1-\frac{\tilde{T}}{\overline{T}}-\frac{\tilde{\rho}}{\overline{\rho}}\right)+\nu\frac{1}{\overline{T}Z}\frac{\partial}{\partial t}\right.$$

$$\times\left\{\left[\frac{\gamma\theta_1}{Z}\frac{U_1}{\overline{V}_1}\left(\frac{1}{\Pr}-\frac{\overline{V}_1}{\overline{V}_1+\overline{C}}\times\frac{\frac{4}{3}-\frac{\gamma-1}{\Pr}}{2}\right)\right.\right.$$

$$\left.-\,\overline{T}\left(\frac{\gamma-1}{\overline{V}+\overline{C}}+\frac{1}{\overline{V}_1}\right)\right]\left[(\overline{T}-U_1)\tilde{V}_1+\overline{V}_1\tilde{T}+\overline{V}_1\overline{T}\frac{\tilde{\rho}}{\overline{\rho}}\right]$$

$$+\left\{\frac{\overline{T}}{\overline{V}_1}+U_1\left[\frac{\left(\frac{4}{3}+\frac{\gamma+\frac{\overline{C}^2}{\overline{V}_1^2}}{\Pr}\right)\theta_1-2\overline{V}_1}{\theta_1\frac{4}{3}\frac{\gamma}{\Pr}(\theta_1-\theta_2)}+\frac{1}{\overline{V}_1}\frac{\frac{4}{3}(\theta_1+\theta_2)-2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)}-\frac{\gamma}{\Pr}\frac{\theta_1}{\overline{V}_1Z}\right.\right.$$

$$\left.\left.+\,\frac{\overline{C}}{\overline{T}Z\overline{V}_1}\left(U_1\frac{\frac{4}{3}(\theta_1+\theta_2)-2\overline{V}_1}{\frac{4}{3}(\theta_1-\theta_2)}+\overline{T}\right)\right]\right\}$$

$$\left.\left.\times\left[\gamma\overline{T}\tilde{V}_1-\overline{C}\tilde{T}-\overline{C}\overline{T}\frac{\tilde{\rho}}{\overline{\rho}}\right]+\frac{Z\overline{T}}{\overline{V}_1}\left(\overline{T}-\frac{\gamma}{\Pr}\frac{\theta_1U_1}{Z}\right)\frac{\tilde{\rho}}{\overline{\rho}}\right\}\right\}-\frac{\gamma\theta_1\overline{V}_1U_1}{Z}\frac{\partial}{\partial x_2}\tilde{V}_2$$

$$+\,\nu\frac{\partial^2}{\partial x_2\partial t}\left[(A_{31})_{11}\tilde{V}_1+(A_{32})_{11}\tilde{V}_2+(A_{33})_{11}\tilde{T}+(A_{34})_{11}\frac{\tilde{\rho}}{\overline{\rho}}\right].\tag{3.31c}$$

Remark 3.2. The expression of matrices A_{00}, A_{01}, A_{10}, B_{00}, B_{01}, and B_{10} is necessary for the analysis of the well-posedness of the initial boundary value problems (see [23]) but the use of relations (3.22)–(3.25) considerably simplifies their implementation and reduces the risks of errors.

3.5. *Absorbing Boundary Conditions for the Linearized Compressible Euler Equations*

They have been obtained following the lines drawn by Engquist and Majda in [7]. It can be easily verified that setting ν to 0 in the artificial boundary conditions of order (1, 0) (resp. (1, 1)) gives the absorbing boundary conditions of order 0 (resp. 1). In this respect, our boundary conditions are continuous in ν as ν tends to zero.

—*Absorbing boundary conditions of order 0:*

*Supersonic inflow,

$$\frac{1}{2\overline{\overline{C}}}\tilde{V}_1 + \frac{1}{2\gamma\overline{\overline{T}}}\tilde{T} + \frac{1}{2\gamma\overline{\rho}}\tilde{\rho} = 0 \tag{3.32a}$$

$$\tilde{V}_2 = 0 \tag{3.32b}$$

$$\frac{1}{\gamma\overline{\overline{T}}}\tilde{T} - \frac{\gamma-1}{\gamma\overline{\rho}}\tilde{\rho} = 0 \tag{3.32c}$$

$$\frac{1}{2\overline{\overline{C}}}\tilde{V}_1 - \frac{1}{2\gamma\overline{\overline{T}}}\tilde{T} - \frac{1}{2\gamma\overline{\rho}}\tilde{\rho} = 0 \tag{3.32d}$$

which is equivalent to $\tilde{V}_1 = \tilde{V}_2 = \tilde{T} = \tilde{\rho} = 0$;

*Subsonic inflow, (3.32b), (3.32c), (3.32d);

*Subsonic outflow, (3.32d);

*Supersonic outflow, no boundary condition.

—*Absorbing boundary conditions of order 1:*

*Supersonic inflow, $\tilde{V}_1 = \tilde{V}_2 = \tilde{T} = \tilde{\rho} = 0$;

*Subsonic inflow, (3.32c)

$$\tilde{V}_1 - \frac{\overline{C}\tilde{T}}{\gamma - 1\,\overline{\overline{T}}} = 0 \tag{3.33}$$

$$\frac{\partial}{\partial t}\tilde{V}_2 + \frac{\overline{C}(\overline{V}_1 + \overline{C})}{(\gamma-1)\overline{T}}\frac{\partial}{\partial x_2}\tilde{T} = 0; \tag{3.34}$$

*Subsonic outflow,

$$\frac{\partial}{\partial t}\left(\frac{1}{2\overline{\overline{C}}}\tilde{V}_1 - \frac{1}{2\gamma\overline{\overline{T}}}\tilde{T} - \frac{1}{2\gamma\overline{\rho}}\tilde{\rho}\right) - \frac{\overline{V}_1}{2\,\overline{C}}\frac{\partial}{\partial x_2}\tilde{V}_2 = 0; \tag{3.35}$$

*Supersonic outflows, no boundary condition.

Remark 3.3. The mixed problem associated with the absorbing boundary conditions of order (0, 0) is proved to be well posed by energy estimates in [23] and by normal mode analysis in [10]. It has been shown in [23] that for the 1D case, the error between its solution and the restriction to $\mathbb{R}_-$ of the solution of the Cauchy problem tends to 0 as ν when ν tends to 0. For the absorbing boundary conditions of order (1, 0) and (1, 1), the associated mixed problems are shown to be well posed when certain terms are neglected (see [23]).

4. DEPENDENCE WITH RESPECT TO THE VISCOSITY: THE 1D CASE

We have seen in Section 2 that the transparent boundary condition (2.9)–(2.10) was approximated first with respect to parameter $\nu \ll 1$ and then with respect to parameter $\varepsilon = i\eta/s \ll 1$. The interest of the one-dimensional case is that the absence of ε allows for a rigorous study of the effects of the approximation with respect to ν.

4.1. *Equations and Boundary Conditions*

The equations reduce to

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} - \nu P\frac{\partial^2 u}{\partial x^2} = 0 \tag{4.1}$$

with

$$u = \begin{pmatrix} \tilde{V} \\ \tilde{T} \\ \dfrac{\tilde{\rho}}{\overline{\rho}} \end{pmatrix}, \quad A = \begin{pmatrix} \overline{V} & R & R\overline{T} \\ (\gamma-1)\overline{T} & \overline{V} & 0 \\ 1 & 0 & \overline{V} \end{pmatrix}, \quad P = \operatorname{diag}\left(\frac{4}{3}, \frac{\gamma}{\mathrm{Pr}}, 0\right).$$

When $\nu = 0$, Eq. (4.1) expresses the transport of the characteristic variables w associated to matrix A. Because of their physical meaning, the w variables will be preferred to the u variables. They are defined by $w = \mathscr{P}^{-1}u$ with

$$\mathscr{P} = \begin{pmatrix} \overline{C} & 0 & \overline{C} \\ (\gamma-1)\overline{T} & \overline{T} & -(\gamma-1)\overline{T} \\ 1 & -1 & -1 \end{pmatrix}, \quad \mathscr{P}^{-1} = \begin{pmatrix} \dfrac{1}{2\overline{\overline{C}}} & \dfrac{1}{2\gamma\overline{\overline{T}}} & \dfrac{1}{2\gamma} \\ 0 & \dfrac{1}{\gamma\overline{\overline{T}}} & -\dfrac{\gamma-1}{\gamma} \\ \dfrac{1}{2\overline{\overline{C}}} & -\dfrac{1}{2\gamma\overline{\overline{T}}} & -\dfrac{1}{2\gamma} \end{pmatrix}$$

and are solutions of the equation

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} - \nu B\frac{\partial^2 w}{\partial x^2} = 0 \tag{4.2}$$

with

$$\Lambda = \mathscr{P}^{-1} A \mathscr{P} = \operatorname{diag}(\lambda_1 = \overline{V} + \overline{C}, \lambda_2 = \overline{V}, \lambda_3 = \overline{V} - \overline{C})$$

and

$$B = \mathscr{P}^{-1} P \mathscr{P} = \begin{pmatrix} \frac{2}{3} + \frac{\gamma - 1}{2\mathrm{Pr}} & \frac{1}{2\mathrm{Pr}} & \frac{2}{3} - \frac{\gamma - 1}{2\mathrm{Pr}} \\ \frac{\gamma - 1}{\mathrm{Pr}} & \frac{1}{\mathrm{Pr}} & -\frac{\gamma - 1}{\mathrm{Pr}} \\ \frac{2}{3} - \frac{\gamma - 1}{2\mathrm{Pr}} & -\frac{1}{2\mathrm{Pr}} & \frac{2}{3} + \frac{\gamma - 1}{2\mathrm{Pr}} \end{pmatrix}.$$

The computational domain is the segment [0, 1] and we restrict ourselves to the case $0 < \overline{V} < \overline{C}$, where both the inflow and outflow boundaries are of subsonic type. This case is more complex than the supersonic case $\overline{C} < \overline{V}$ because information (w_3) propagates against the flow.

The one-dimensional boundary conditions can be obtained by removing the terms in $\tilde{V}_2$ in the corresponding two-dimensional boundary conditions and replacing $\tilde{V}_1$, $\overline{V}_1$, and x_1 by $\tilde{V}$, $\overline{V}$ and x, respectively. Their expression in terms of variables w is given in [23].

The boundary conditions at $x = 0$ are derived by replacing x by $-x$, $\overline{V}$ by $-\overline{V}$, and $\tilde{V}$ by $-\tilde{V}$ in the conditions corresponding to the negative half-space in the subsonic inflow case. We have three conditions at $x = 0$ and two at $x = 1$.

The artificial boundary conditions will be compared to the transparent boundary conditions for the one-dimensional Euler equations: $w_1 = w_2 = 0$ at $x = 0$ (two boundary conditions) and $w_3 = 0$ at $x = 1$ (one boundary condition) and to the Gustafsson and Sundström boundary conditions we recall below. In [8], they used energy estimates to derive boundary conditions leading to well-posed initial boundary value problems and such that as ν tends to 0 the corresponding hyperbolic problems are also well posed. Their boundary conditions read at $x = 0$:

- $w_1 = 0$
- $w_2 = 0$,

$$\nu \frac{\partial}{\partial x}\left[\left(\frac{4}{3} - \frac{\gamma - 1}{\mathrm{Pr}}\right) w_1 - \frac{1}{\mathrm{Pr}} w_2 + \left(\frac{4}{3} + \frac{\gamma - 1}{\mathrm{Pr}}\right) w_3\right] = 0;$$

and at $x = 1$:

$$\overline{C} w_3 + \frac{2}{3} \nu \frac{\partial}{\partial x}(w_1 + w_3) = 0$$

$$\nu \frac{\partial}{\partial x}[(\gamma - 1)(w_1 - w_3) + w_2] = 0.$$

For details, see [8] and also [23].

4.2. *The Numerical Scheme*

In order to approximate the hyperbolic part of the equations in the most suitable way, we have chosen to use the algorithm introduced by Zalesak in [24] for multidimensional nonlinear hyperbolic systems. This algorithm adds corrected antidiffusive fluxes to the numerical fluxes of a monotonous first-order scheme in such a way that local extrema are not created nor increased. The first-order scheme is the explicit upwind scheme and the high order scheme is Lax–Wendroff scheme.

4.3. *Discretization of the Artificial Boundary Conditions*

The segment [0, 1] is divided into I intervals $[x_i, x_{i+1}]$, $0 \le i \le I-1$ with $I = 1/\Delta x$ and $x_i = i\Delta x$. We have chosen $\Delta x = 10^{-2}$, i.e. $I = 100$. The scheme is applied from x_0 to x_I and we introduce four ficticious points x_{-2}, x_{-1}, x_{I+1}, and x_{I+2}. At points x_{-1} and x_{I+1}, we apply the scheme with simplified limitation coefficients and the boundary conditions are discretized at x_{-2} and x_{I+2}.

4.4. *The Numerical Boundary Conditions*

At $x = 0$ (subsonic inflow), the artificial boundary conditions are a number of 3 (see Section 3.2). In all other cases, the number of discretized continuous boundary conditions is less than or equal to two and it is necessary to introduce extra relations, the so-called "numerical boundary conditions," in order to get a system of three equations for the three unknowns that are components of vector w_{-2}^{n+1} or w_{I+2}^{n+1}.

At x_{-2}, since w_3 propagates in the negative x direction and since we assume $\nu \ll 1$, we use an upwind discretization of advection equation

$$\frac{\partial}{\partial t} w_3 + (\overline{V} - \overline{C}) \frac{\partial}{\partial x} w_3 = 0.$$

At x_{I+2}, we again use upwind discretizations of advection equations

$$\frac{\partial}{\partial t} w_1 + (\overline{V} + \overline{C}) \frac{\partial}{\partial x} w_1 = 0, \tag{4.3}$$

$$\frac{\partial}{\partial t} w_2 + \overline{V} \frac{\partial}{\partial x} w_2 = 0. \tag{4.4}$$

4.5. *Numerical Results*

We choose $\overline{V} = 1$, $\overline{\rho} = 1$, and $\overline{C} = 2$ and the classical values $\gamma = 1.4$ and $\mathrm{Pr} = 0.75$. We also take $R = 1$ because we assume that the equations have been *nondimensionalized*. As $\overline{C}$ is related to $\overline{T}$ by $\overline{C} = (\gamma R \overline{T})^{1/2}$, we find $\overline{T} \cong 2.86$. The flow is subsonic and the characteristic variables w_1, w_2, and w_3 propagate at respective speeds $\overline{V} + \overline{C} = 3$, $\overline{V} = 1$, $\overline{V} - \overline{C} = -1$.

Each characteristic variable has initial value

$$f_0(x) = \begin{cases} e^{1/r^2} e^{1/((x-x_C)^2 - r^2)} & \text{if } |x - x_C| < r, \\ 0 & \text{otherwise;} \end{cases}$$

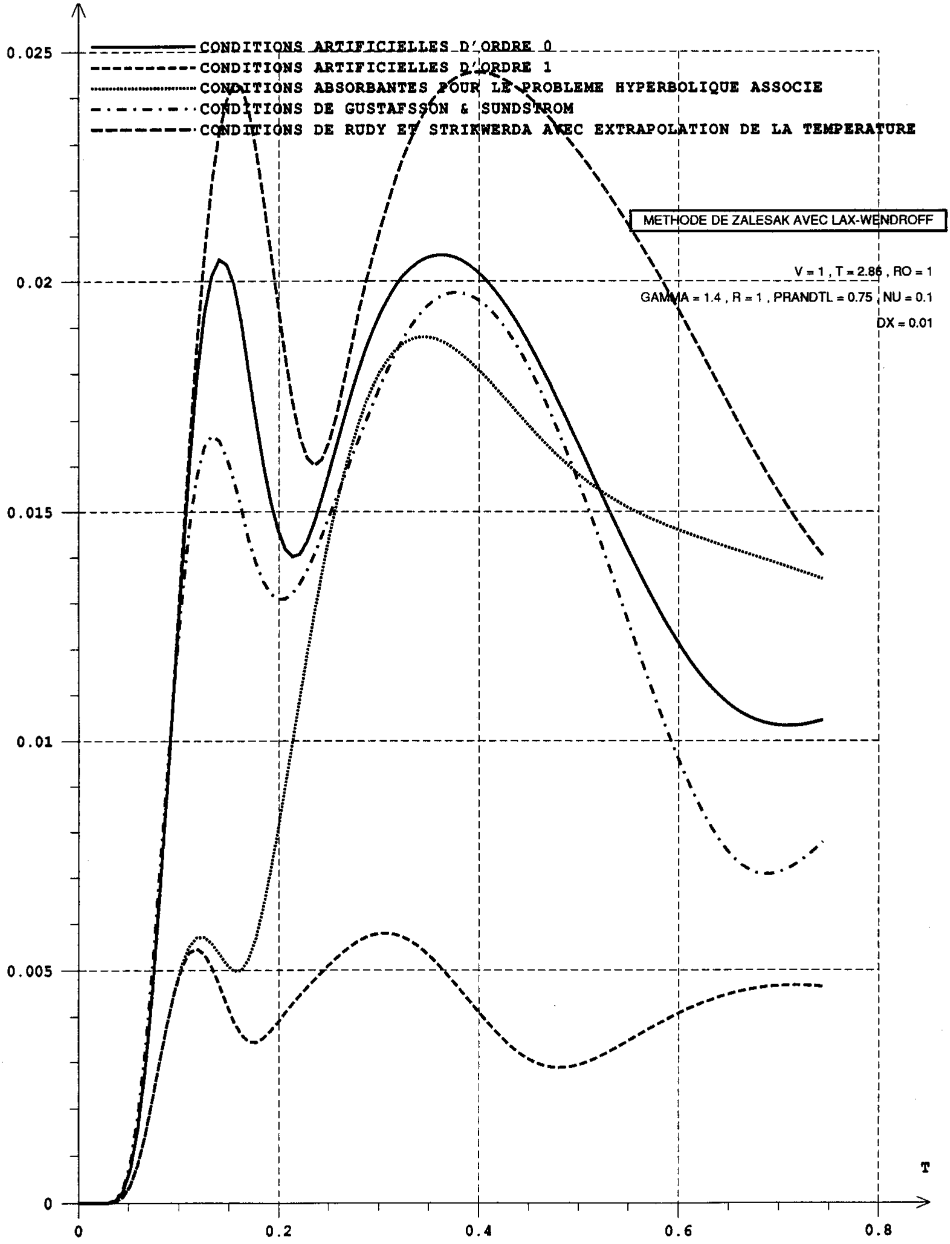

FIG. 1. Time-evolution of the (normalized) l^2-norm of the error between the solution of the Cauchy problem and the solutions of the mixed problems associated to: the artificial boundary conditions of orders 0 (solid line) and 1 (dashed line), the absorbing boundary conditions for the Euler equations (dotted line), Gustafsson and Sundström boundary conditions (dot–dashed line) and Rudy and Strikwerda nonreflecting boundary condition (long-dashed line).

f_0 belongs to $C^\infty(\mathbb{R})$ and has compact support in the bowl centered around x_c with radius r. We choose $x_c = \frac{1}{2}$, $r = \frac{1}{4}$.

Figure 1 shows the l^2-norm of the error between the solution of the discretized

Cauchy problem ($\nu = 0.1$) and the solution of the discretized initial boundary value problems corresponding to the different boundary conditions, divided by the l^2-norm of the initial value. The solution of the Cauchy problem results from a computation on an interval $[-L, 1 + L]$ with L "sufficiently" large (see [23] for details).

Although Gustafsson and Sundström's boundary conditions have not been introduced in order to be used as artificial boundary conditions, they give good results. This is probably due to the fact that they by construction, contain the absorbing boundary conditions for the Euler equations.

We also introduce the nonreflecting boundary condition of Rudy and Strikwerda. It is described in detail in [19, 20]. It reads $(\partial/\partial t)p - \rho C(\partial/\partial t)V + \alpha(p - \overline{p}) = 0$, the optimal choice for parameter α being

$$\alpha^* = \frac{\overline{C}^2 - \overline{V}^2}{\overline{C}}(\zeta^* - 1), \quad \text{with } \zeta^* = 1.2784645.$$

It has been introduced for the calculation of steady solutions when using a pseudo-unsteady approach.

We approximate it at point x_{I+2} by

$$\frac{\tilde{p}_{I+2}^{n+1} - \tilde{p}_{I+2}^{n}}{\Delta t} - \overline{\rho}\overline{C}\frac{\tilde{V}_{I+2}^{n+1} - \tilde{V}_{I+2}^{n}}{\Delta t} + \alpha\tilde{p}_{I+2}^{n+1} = 0 \tag{4.5}$$

and we complete it according to the strategy proposed in [20]:

—zeroth-order extrapolation of $\tilde{V}$

—zeroth-order extrapolation of $\tilde{T}$

—calculation of $\tilde{p}$ by means of (4.5)

—calculation of $\tilde{\rho}$ using the state law $\overline{p} + \tilde{p} = (\overline{\rho} + \tilde{\rho})R(\overline{T} + \tilde{T})$.

The artificial boundary conditions of order 1 in ν clearly give the lowest level of the error.

The fact that the artificial boundary conditions of order 0 better approximate the transparent boundary condition (2.9)–(2.10) with respect to the parameter ν is not clear for $t \leq 0.525$. This is certainly due to the numerical boundary conditions. More precisely, at $x = x_{I+2}$, for example, the transparent boundary condition for the Euler equations $w_3 = 0$ is completed by upwind discretizations of transport equations (4.3) and (4.4), whereas the two discretized artificial boundary conditions of order 0 are completed by an upwind discretization of (4.4). The set of the two artificial boundary conditions of order 0 seems to be less efficient, at least for the short times, than the one made up of conditions $w_3 = 0$ and (4.3). An explanation for this phenomenon will be proposed at the end of next section. The error associated to the artificial boundary conditions of order 1 is the smallest.

Figure 2 shows the asymptotic behaviour of the error when the parameter ν tends to zero, whereas Fig. 3 indicates how fast the error diminishes. Between the artificial boundary conditions of order 0 and 1, the slopes have a ratio greater than 2, and the slope corresponding to the transparent boundary conditions for the Euler equations is slightly lower than the one related to the conditions of order 0.

Remark 4.1. In order to maintain negligible numerical damping terms in front of the physical diffusion terms, an analysis of the differential equation equivalent to the scheme at first order in time and second order in space shows that we must ensure $\nu/\Delta x \gg \| |\Lambda| \|/2\|\mathrm{B}\|$, where $\|\cdot\|$ denotes a matrix norm [23].

5. DEPENDENCE WITH RESPECT TO THE ANGLE OF INCIDENCE: THE 2D CASE

In this section, the effects of approximating the transparent boundary condition (2.9)–(2.10) with respect to variable $\varepsilon = i\eta/s$, will be numerically analyzed through a model problem. We will work with the physical variables $u = (\widetilde{V_1}, \widetilde{V_2}, \tilde{T}, \tilde{\rho}/\overline{\rho})^t$ and the space coordinates will be denoted (x, y), instead of (x_1, x_2).

5.1. *The Model Problem*

We want to solve the linearized 2D Navier–Stokes equations on the strip $\mathbb{R} \times [0, 1]$ of xOy plane.

At $x = 0$ and at $x = 1$, we introduce artificial boundaries where we successively adopt

—the absorbing boundary conditions of order 0 for the Euler equations,

—the absorbing boundary conditions of order 1 for the Euler equations,

—the absorbing boundary conditions of order (0, 0),

—the absorbing boundary conditions of order (1, 0),

—the absorbing boundary conditions of order (1, 1).

On the north boundary ($y = 1$), we impose in all cases the absorbing boundary conditions of order 0 for the Euler equations. On the south boundary ($y = 0$), we also employ the absorbing boundary conditions of order 0 for the Euler equations except when $\overline{V}_2 = 0$ and $u(\cdot, t = 0)$ is symmetrical with respect to Ox axis which becomes a symmetry axis.

For quantities $\overline{V}_1, \overline{T}, \overline{\rho}, \gamma, R$, and Pr, we keep the values of the previous section. Moreover, we choose $\overline{V}_2 = 0$. As $\overline{V}_1 < \overline{C}$ and $\overline{V}_2 < \overline{C}$, the flow is subsonic in each space direction. The west boundary is of subsonic inflow type, whereas the east boundary is of subsonic outflow type. Let us introduce the two scalar functions

$$f_0(x, y) = \begin{cases} e^{1/r^2}\, e^{\frac{1}{(x-x_C)^2+(y-y_C)2-r^2}}, & \text{if } (x - x_C)^2 + (y - y_C)^2 < r^2 \\ 0, & \text{otherwise;} \end{cases} \tag{5.1}$$

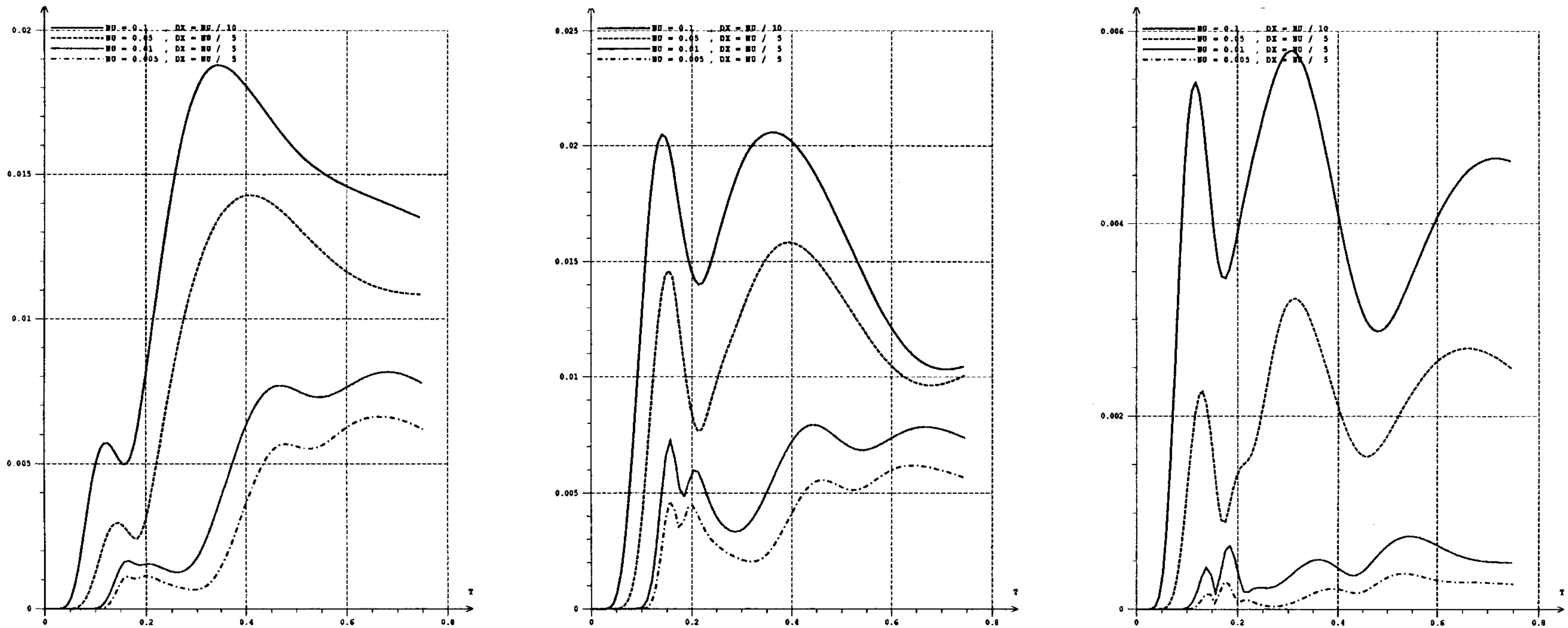

FIG. 2. Asymptotic behaviour of the error as ν tends to 0 for the artificial boundary conditions of orders 0 (center) and 1 (right) and the absorbing boundary conditions for the Euler equations (left). Solid line, $\nu = 0.1$, $\Delta x = \nu/10$; dashed line, $\nu = 0.05$, $\Delta x = \nu/5$; dotted line, $\nu = 0.01$, $\Delta x = \nu/10$; dot-dashed line, $\nu = 0.005$, $\Delta x = \nu/5$, $\overline{V} = 1$, $\overline{T} = 2$, $\overline{\rho} = 1$, $\gamma = 1.4$, $R = 1$, $\Pr = 0.75$.

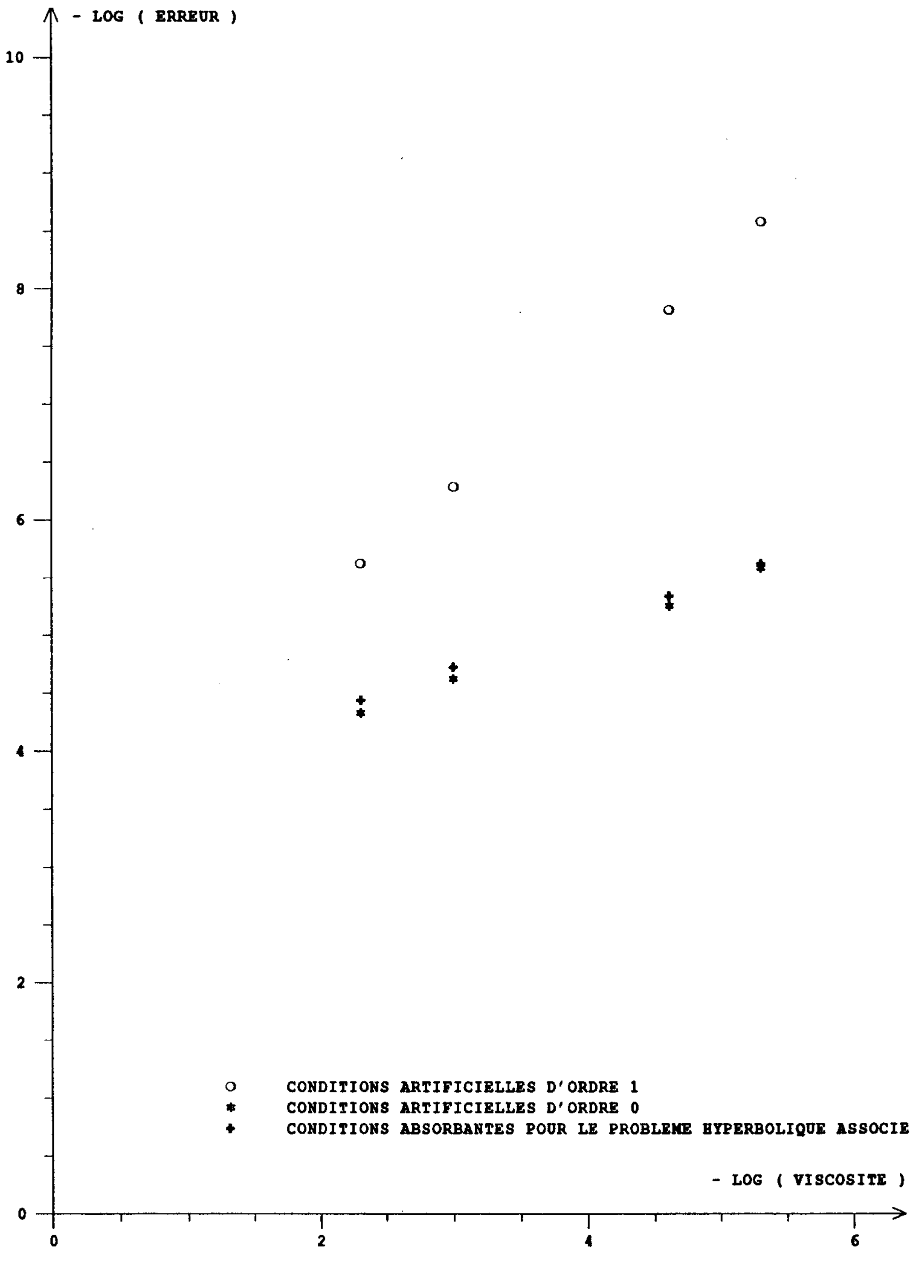

FIG. 3. l^2-norm (space-time) of the error as a function of ν for the artificial boundary conditions of orders 0 (stars) and 1 (circles) and the absorbing boundary conditions for the Euler equations (plus signs), logarithmic scale. $\overline{V} = 1$, $\overline{T} = 2.86$, $\overline{\rho} = 1$, $\gamma = 1.4$, $R = 1$, $\mathrm{Pr} = 0.75$; NU = 0.1, DX = NU/10; NU = 0.05, DX = NU/5; NU = 0.01, DX = NU/5; $\nu = 0.005$, $\Delta x = \mathrm{NU}/5$.

$$g_0(x, y) = f_0(x, y) \cos[k_x(x - x_C) + k_y(y - y_C)], \tag{5.2}$$

where f_0 is infinitely differentiable and has compact support in the bowl centered around point (x_C, y_C) with radius r.

The initial value is defined by $\tilde{V}_2 = 0$ and $\tilde{V}_1 = \tilde{T} = \tilde{\rho}/\overline{\rho} = f_0$, or $\tilde{V}_1 = \tilde{T} = \tilde{\rho}/\overline{\rho} = g_0$. By modifying the direction of wave vector $\mathbf{k} = (k_x, k_y)^t$ in function g_0, we are able to investigate the effects of approximating the transparent boundary condition with respect to the parameter ε.

5.2. *Discretization and Numerical Boundary Conditions*

As for the 1D case, the Euler equations are approximated with Zalesak's fully multidimensional flux corrected transport algorithm [24].

The segment [0, 1] on the Ox axis is divided into I intervals $[x_i, x_{i+1}]$, $0 \leq i \leq I - 1$ with $I = 1/\Delta x$ and $x_i = i\,\Delta x$, whereas the segment [0, 1] on the Oy axis is split into J intervals $[y_j, y_{j+1}]$, $0 \leq j \leq J - 1$ with $J = 1/\Delta y$ and $y_j = j\,\Delta y$. We have chosen $\Delta x = \Delta y = 2 \times 10^{-2}$, i.e. $I = J = 50$.

We introduce the ficticious coordinates $x_{-2}, x_{-1}, x_{I+1}, x_{I+2}, y_{-2}, y_{-1}, y_{J+1}$, and y_{J+2} and we apply the scheme at points (x_i, y_j), $-1 \leq i \leq I + 1$, $-1 \leq j \leq J + 1$.

The north boundary is defined by $-1 \leq i \leq I + 1$ and $j = J + 2$; the south boundary is defined by $-1 \leq i \leq I + 1$ and $j = -2$; the west boundary is defined by $i = -2$ and $-2 \leq j \leq J + 2$; and, finally, the east boundary is defined by $i = I + 2$ and $-2 \leq j \leq J + 2$.

The boundary conditions are written at the second rank of fictitious points at time t_{n+1}.

On the west boundary, for the artificial boundary conditions, x-derivatives are replaced by first-order forward finite differences, y-derivatives are replaced by second-order centered finite differences (except for $j = -2$ and $j = J + 2$, where first-order noncentered finite differences are used) and time derivatives are replaced by first-order backward finite differences.

On the east boundary, for the artifical boundary conditions, x-derivatives are approximated by first-order backward finite differences, whereas y- and t-derivatives are treated in the same way as on the west boundary.

For more details on the discretization of the continuous boundary conditions, see [23].

As for the 1D case, when the number of continuous boundary conditions is less than or equal to 3, the discrete boundary conditions resulting from their approximation have to be completed by numerical boundary conditions in order to get a system of four equations for the four unknowns that are the components of vector $u_{i,j}^{n+1}$ at a boundary node.

Notice that the problem does not arise at the west boundary for the artificial boundary conditions that are a number of four. In all other cases, numerical boundary conditions have to be added.

As a general rule, in the interior of east and west boundaries (resp. north and south), we use the unperturbed ($\nu = 0$) evolution equations of matrix $A^{(1)}$ (resp. $A^{(2)}$) characteristic variables that propagate outside the computational domain, because it is then safe to approximate $x-$ (resp. $y-$) derivatives by finite differences biased towards the interior of the computational domain. At the north (resp. south) corners, we try as much as possible to account for the continuous north boundary conditions (resp. the three continuous south boundary conditions).

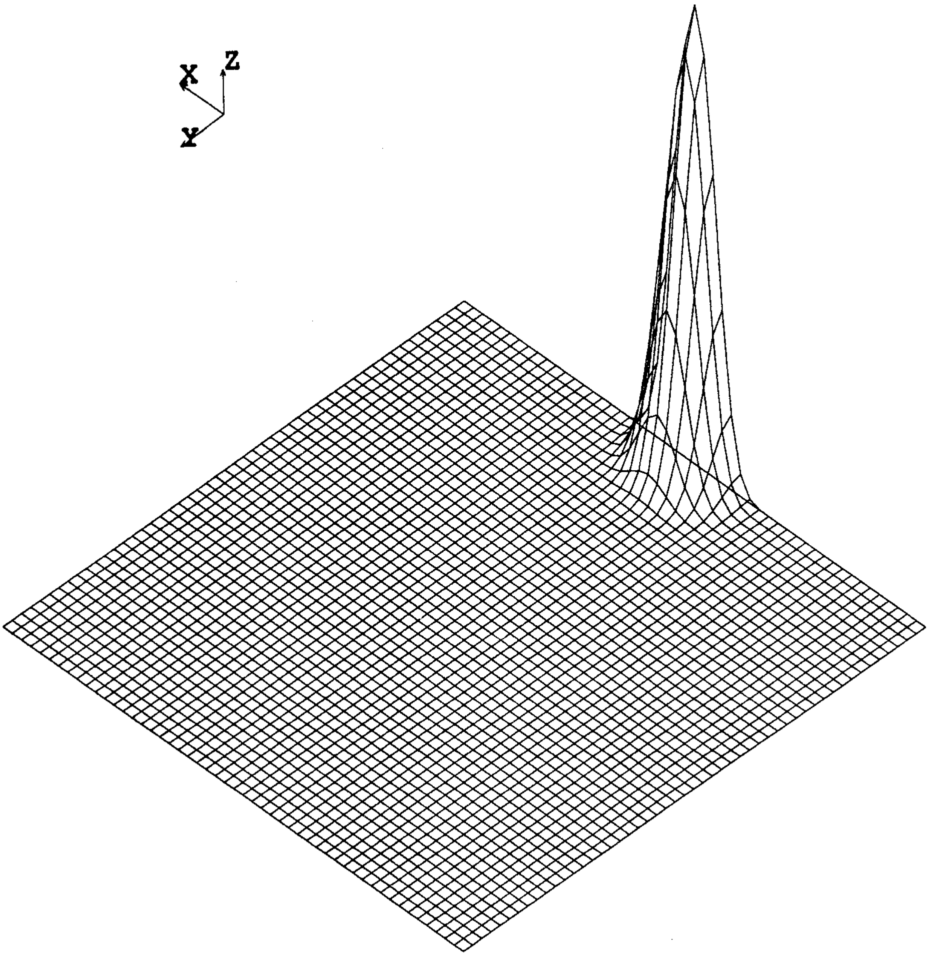

FIG. 4. Initial value f_0 given by (5.1).

The derivation of the numerical boundary conditions as well as their efficient implementation is detailed in [23].

5.3. *A First Set of Numerical Results*

The initial value is given either by function f_0 defined by (5.1) with $x_c = \frac{1}{2}$, $y_c = 0$, and $r = \frac{1}{4}$ (Fig. 4), or by function g_0 defined by (5.2) with $x_c = \frac{1}{2}$, $y_c = \frac{1}{2}$, $r = 0.45$ and $|\mathbf{k}| = 2\pi/10\ \Delta x$. The angle $(\widehat{Ox, \mathbf{k}})$ between axis and wave vector $\mathbf{k}$ having values 0, $\pi/16$, $\pi/8$, or $\pi/4$. Figure 5 shows function g_0 in the case $(\widehat{Ox, \mathbf{k}}) = \pi/4$.

In Fig. 6, we have superimposed, as a function of time, the relative l^2-norm of the error between the solution of the Cauchy problem and the solutions of the initial boundary value problems corresponding to the artificial boundary conditions of order (0, 0), (1, 0), (1, 1) and also to the absorbing boundary conditions of order 0 and 1 for the Euler equations. The initial value is given by function f_0. We obtain a great improvement when we switch from order (0, 0) to (1, 0). For $t \leq 0.26$, the

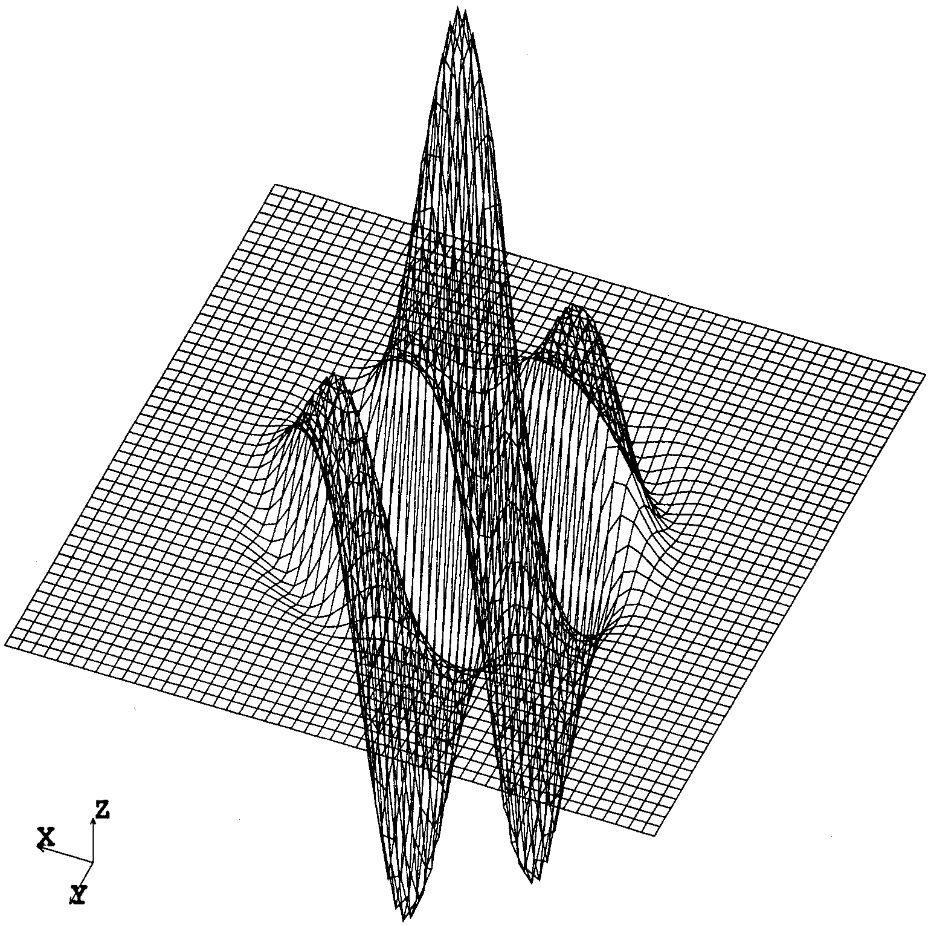

FIG. 5. Initial value g_0 given by (5.2) in the case $(\widehat{Ox, \mathbf{k}}) = \pi/4$.

conditions of order (1, 1) give the best results but surprisingly, they produce for $t \geq 0.29$ the greatest error level. This point will be investigated further in 5.4. For $t \geq 0.19$, the absorbing boundary conditions of order 0 correspond to the greatest error level whereas for $t \geq 0.24$, the absorbing boundary conditions of order 1 appear to be the best of all.

Figure 7 shows the influence of the angle $(\widehat{Ox, \mathbf{k}})$ in g_0 for the artificial boundary conditions of order (1, 1). For the long times, we see that the error grows with the angle, what could be expected from a Taylor expansion in the vicinity of $\varepsilon = i\eta/s = 0$.

5.4. *Improvement of the Artificial Boundary Conditions of Order* (1, 1)

We may think that in the artificial boundary conditions of order (1, 1), the Euler equations have not been taken into account in an optimal way when approximating the transparent boundary condition (2.9)–(2.10). It is possible, however, to obtain an equivalent formulation of condition (2.9) where the Euler part explicitly appears.

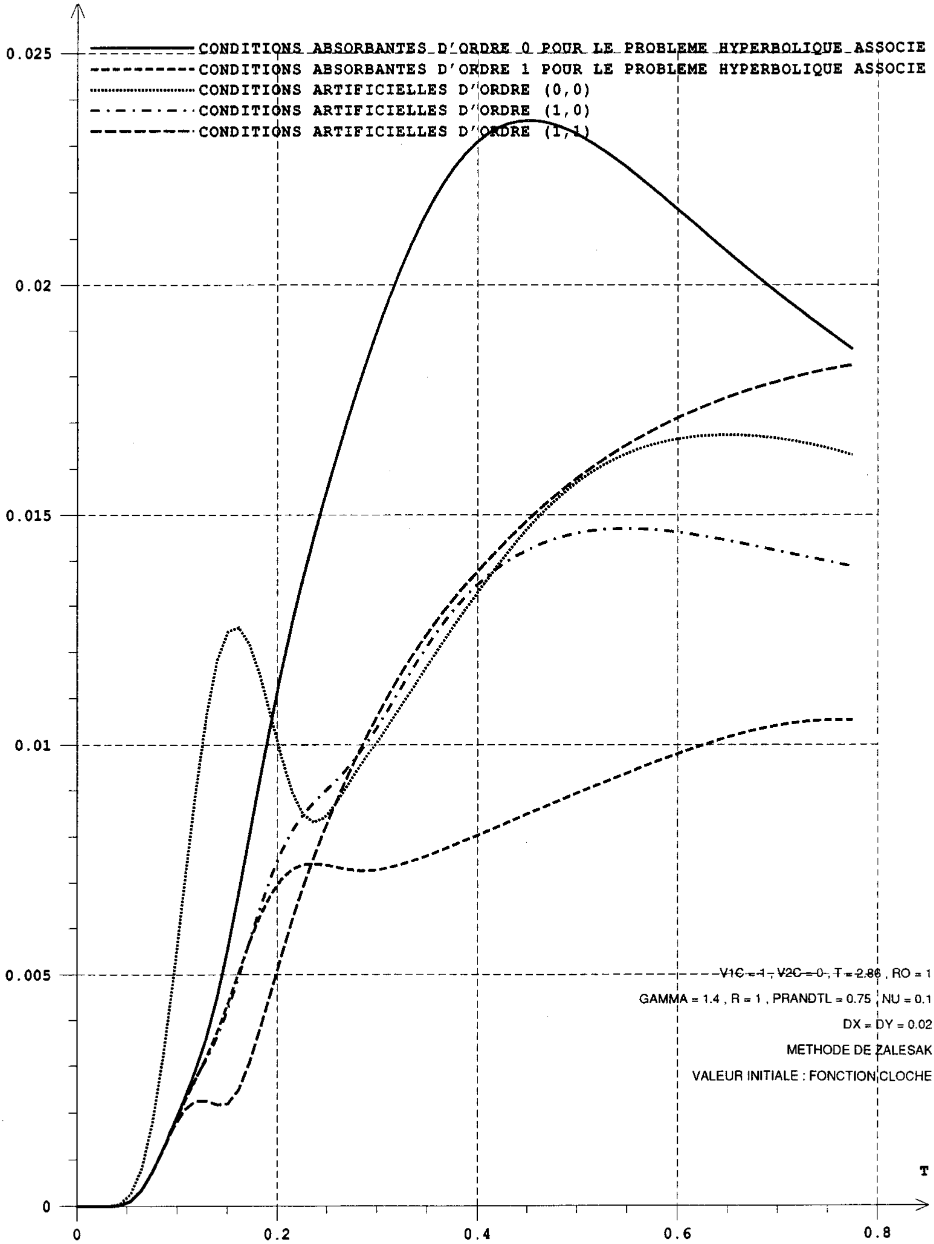

FIG. 6. Time-evolution of the l^2-norm of the error associated to the artificial boundary conditions of orders (0, 0) (dotted line), (1, 0) (dot-dashed line), (1, 1) (long-dashed line) and to the absorbing boundary conditions of orders 0 (solid line) and 1 (dashed line) for the Euler equations.

PROPOSITION 5.1. *The transparent boundary condition* (2.9) *is equivalent to*

$$\nu \frac{d}{dx_1} \widehat{u_-}^I(x_1 = 0) = \Bigg\{ (I - \varepsilon \nu s Q(1,2))^{-1} \Bigg[\nu s (B^{(1)} + \varepsilon B^{(2)} + \varepsilon^2 \nu s Q^{(2,2)}) \widehat{u_-}(x_1 = 0) + Q^{(1,1)} \sum_{j=1}^{r+p} (\widehat{u_-})_j (x_1 = 0) \sum_{i=1}^{r+p} M_{ij}^{-1} (\nu \xi_i)^2 \Phi^i \Bigg] \Bigg\}^I \tag{5.3}$$

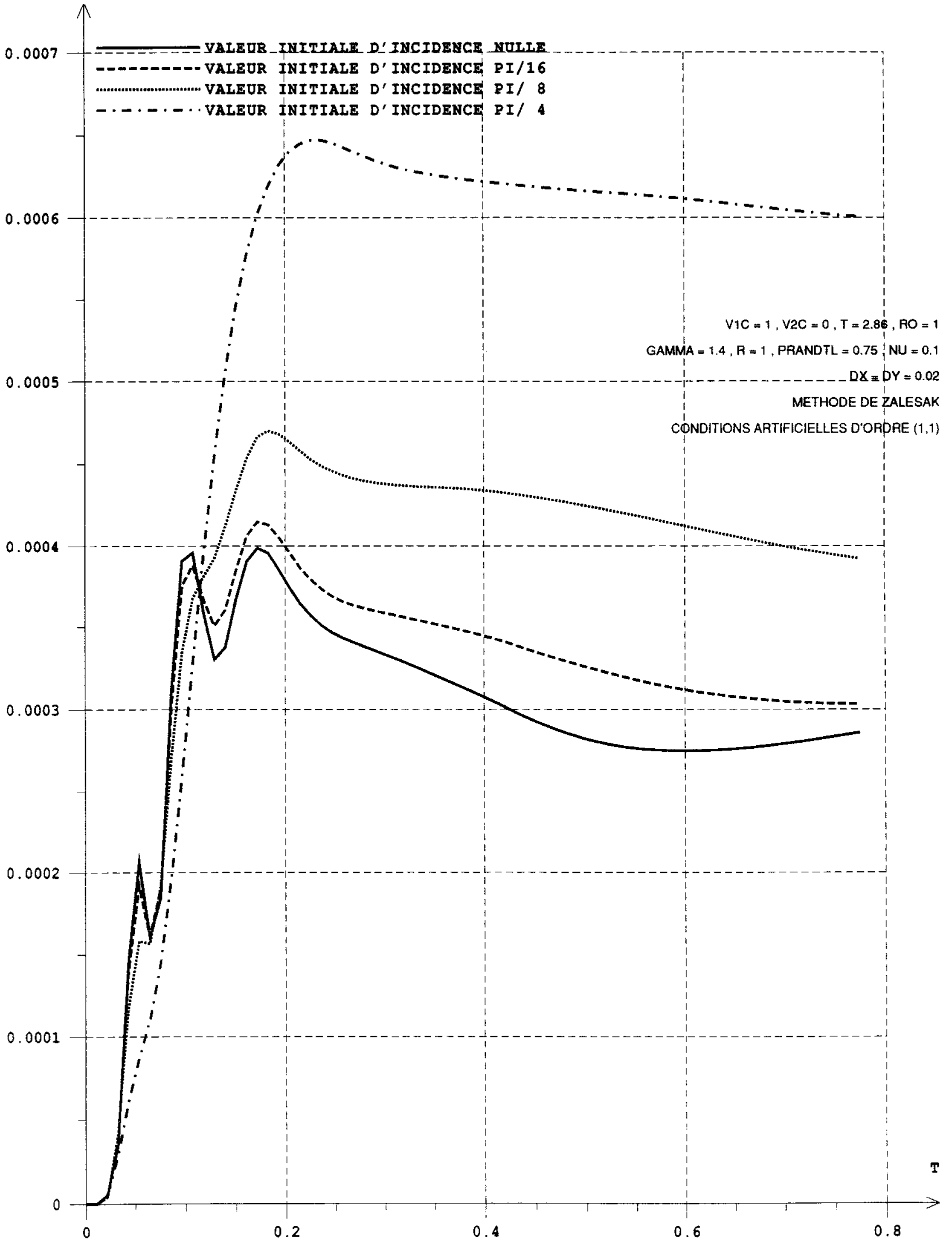

FIG. 7. Time-evolution of the l^2-norm of the error associated to the artificial boundary conditions of order (1, 1) for $(\widehat{Ox, \mathbf{k}}) = 0$ (solid line), $\pi/16$ (dashed line), $\pi/8$ (dotted line), and $\pi/4$ (dot-dashed line) in function g_0.

with

$$B^{(1)} = A^{(1)-1}, \quad B^{(2)} = -B^{(1)}A^{(2)}, \quad Q^{(1,1)} = -B^{(1)}P^{(1,1)},$$
$$Q^{(1,2)} = -2B^{(1)}P^{(1,2)}, \quad Q^{(2,2)} = -B^{(1)}P^{(2,2)}.$$

The proof of this proposition as well as the expressions of matrices $B^{(1)}$, $B^{(2)}$, $Q^{(1,1)}$, $Q^{(1,2)}$ =, and $Q^{(2,2)}$ are given in [23].

Condition (5.3) can then be approximated to the order (1, 1) with respect to (ν, ε). In the resulting boundary condition, the terms $(B^{(1)}\widehat{u_-})^I$ and $(B^{(2)}\widehat{u_-})^I$ have not undergone any approximation, which was not the case with the "old" artificial boundary conditions of order (1, 1). Many more details can be found in [23].

Figure 8 shows that the improved artificial boundary conditions of order (1, 1) now give the lowest error level.

5.5. *Numerical Boundary Conditions: A New Approach*

It comes out from Proposition 5.1 that the transparent boundary condition (2.9) is another formulation for variables $\widetilde{V_1}$, $\widetilde{V_2}$, and $\tilde{T}$ evolution equations at $x_1 = 0$. We have seen in Section 5.2 that in the subsonic outflow case, the three artificial boundary conditions must be completed by a numerical boundary condition. The above interpretation naturally suggests the continuity equation. In terms of Fourier–Lapace variables, it reads

$$\nu s\left(\frac{\hat{\tilde{\rho}}}{\overline{\rho}}\right)_- + \overline{V}_1 \nu \frac{d}{dx_1}\left(\frac{\hat{\tilde{\rho}}}{\overline{\rho}}\right)_- = -\nu \frac{d}{dx_1}(\hat{\tilde{V}}_1)_- - \varepsilon \nu s\left[(\hat{\tilde{V}}_2)_- + \overline{V}_2\left(\frac{\hat{\tilde{\rho}}}{\overline{\rho}}\right)_-\right]. \tag{5.4}$$

We can replace $\nu(d/dx_1)(\hat{\tilde{V}}_1)_-$ using (5.3) and approximate the resulting condition to the desired order with respect to (ν, ε). In the subsonic outflow case, $\overline{V_1} > 0$ and operator $\partial/\partial t + \overline{V_1}(\partial/\partial x_1)$ expresses transport of variable $\tilde{\rho}/\overline{\rho}$ in the positive x-direction. Thus, it is quite safe to approach $\nu(d/dx_1)(\hat{\tilde{\rho}}/\overline{\rho})$ by an upwind finite difference.

It is also possible to generalize the numerical boundary conditions of inviscid type used so far to the case where $\nu \neq 0$. Using Eqs. (5.3) and (5.4) and making an approximation to the order (1, 1) with respect to (ν, ε), the linearized Navier–Stokes equations can be written in terms of Fourier–Laplace variables in the general form

$$\nu \frac{d}{dx_1}\hat{u}(x_1 = 0) - (\mathbf{B}_{00} + \varepsilon\mathbf{B}_{01} + \nu s\mathbf{B}_{10} + \nu s\varepsilon\mathbf{B}_{11})\hat{u}(x_1 = 0) = 0. \tag{5.5}$$

Multiplying (5.5) on the left by each left eigenvector of matrix $A^{(1)}$ corresponding to a positive eigenvalue $\lambda_i^{(1)}$, we obtain boundary conditions in which the normal derivative of the corresponding characteristic variable w_i can safely be approximated by an upwind finite difference. The expressions of these more rigorous numerical boundary conditions are given in [23]. Used together with the improved artificial boundary conditions of order (1, 1), they appeared to give quite good results (Fig. 9). The new approach does not actually provide for any big advantage in terms of the error reduction, as can be seen by comparing Figs. 8 and 9, but the important point is that it is much more satisfying from the mathematical point of view.

6. CONCLUSION

High order artificial boundary conditions have been derived for the compressible Navier–Stokes equations linearized about a constant state using the Fourier and

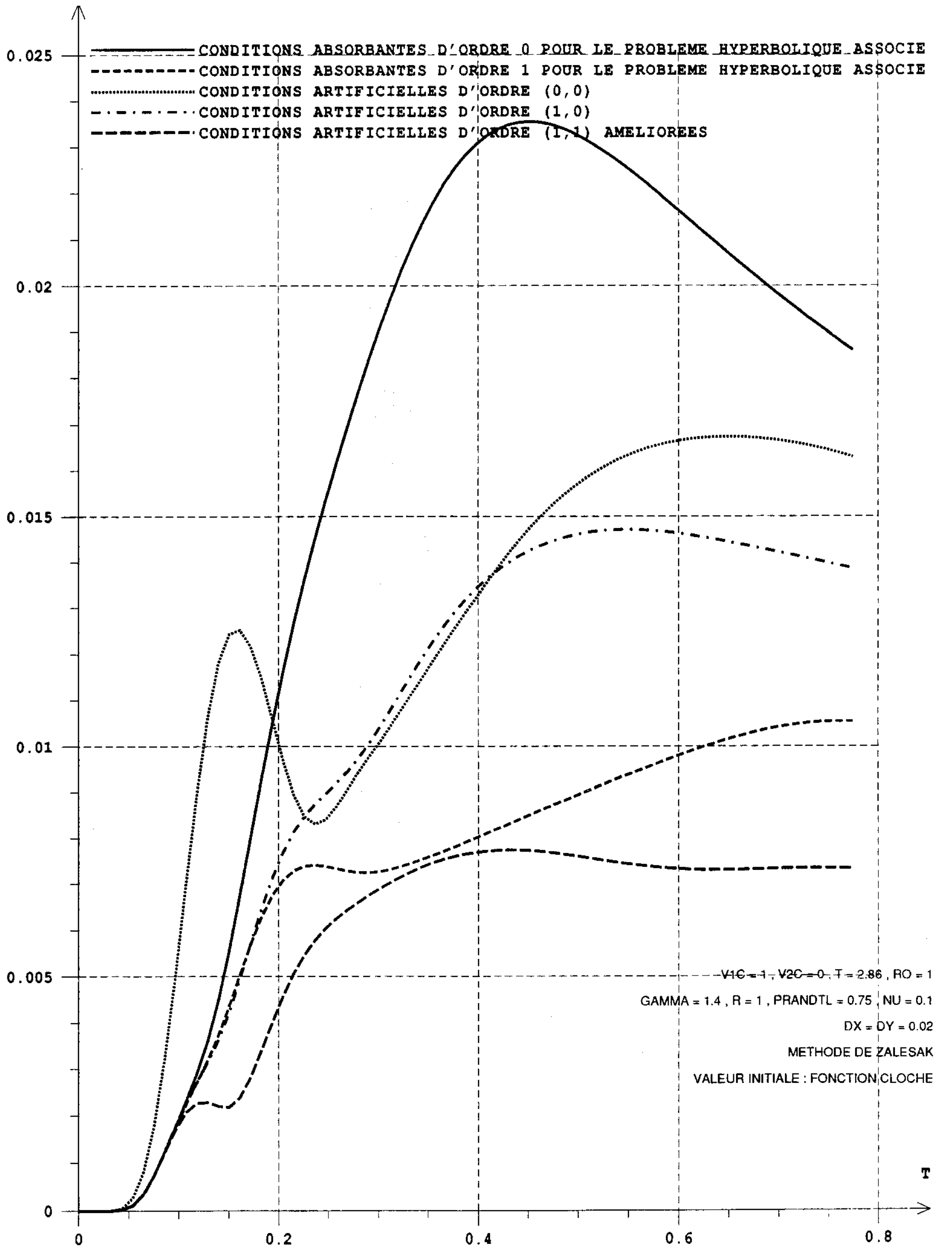

FIG. 8. Time-evolution of the l^2-norm of the error associated to the *improved* artificial boundary conditions of order (1, 1) (long-dashed line), compared to the artificial boundary conditions of orders (0, 0) (dotted line) and (1, 0) (dot-dashed line), and the absorbing boundary conditions of order 0 (solid line) and 1 (dashed line) for the Euler equations.

Laplace transforms and asymptotic expansions under the assumption of small viscosity, high time frequencies and long space wavelengths. They have been implemented in 1D and 2D model problems and compared to the most commonly used artificial boundary conditions (absorbing boundary conditions for the Euler equations, Gus-

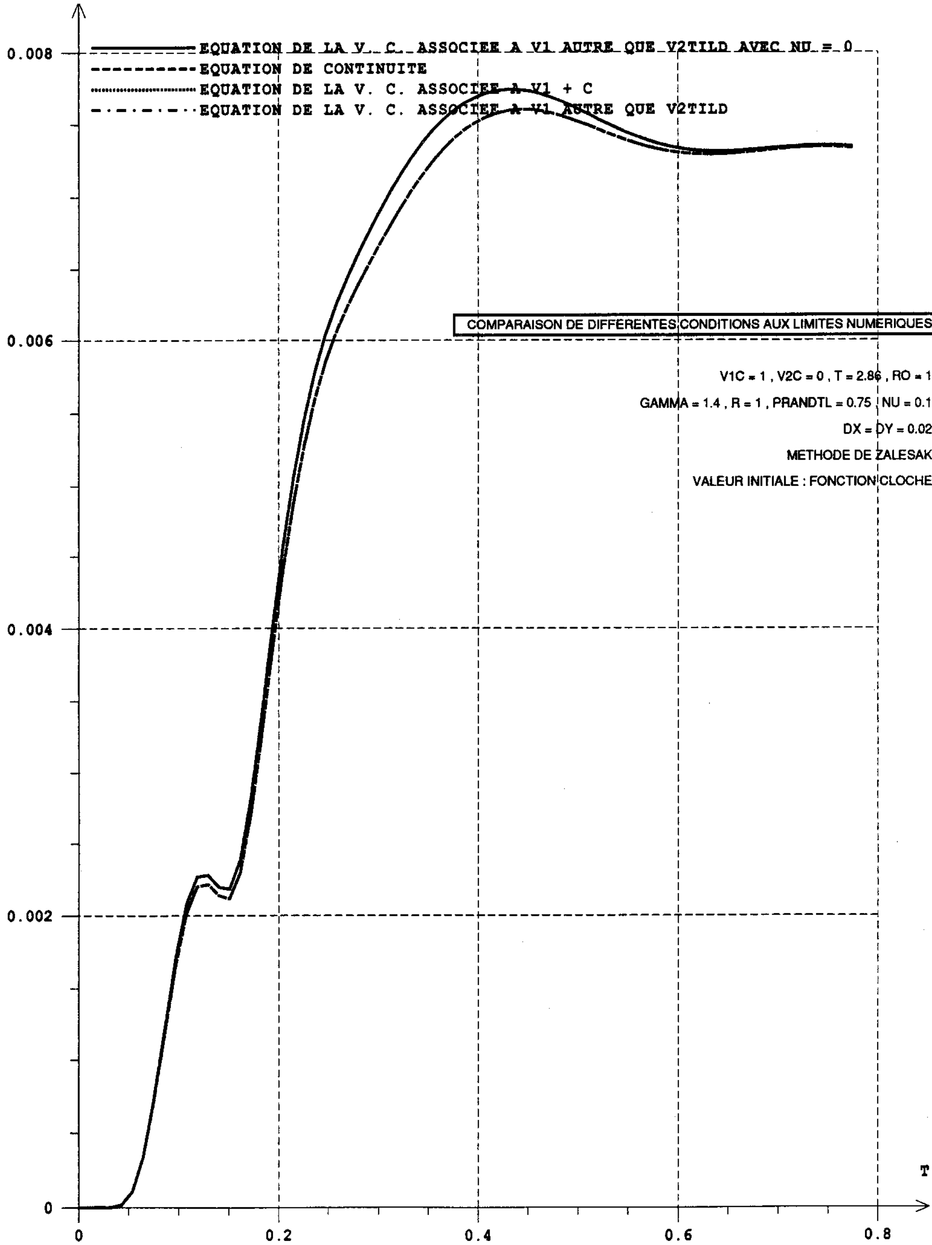

FIG. 9. Time-evolution of the l^2-norm of the error associated to the *improved* artificial boundary conditions of order (1, 1) for several choices of the "numerical boundary condition," showing that the continuity equation (dashed line) performs quite well. Solid line: equation of the characteristic variable associated to $\overline{V}_1$ other than $\tilde{V}_2$, with $\nu = 0$, dotted line: equation of the characteristic variable associated to $\overline{V}_1 + \overline{C}$, dot-dashed line: equation of the characteristic variable associated to $\overline{V}_1$ other than $\tilde{V}_2$.

tafsson and Sundström dissipative boundary conditions and Rudy and Stikwerda nonreflecting boundary condition). The "improved" artificial boundary conditions of order $(1, 1)$ provide the best results. At the discretization level, it may be necessary to introduce extra relations to close the set of artificial boundary conditions and we have proposed a rigorous method for their definition.

There is another approach to the problem of artificial boundary conditions. Introduced for the wave equations by Engquist and Majda and also by Halpern, it consists in working directly on the discretized equations (i.e., on a scheme). This approach has been successfully applied by the author to the linearized compressible Navier–Stokes equations in [23]. The asymptotic expansions with respect to the viscosity are replaced by developments under the assumption of low time frequencies. Moreover, the method directly provides the right number of boundary conditions. The results of this work will hopefully be presented in a forthcoming article.

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